Math 481: Homework 2

Due Wednesday, February 12, 2020

1. (a) Propose a smooth atlas for the set $S^1 \times S^1$.

   (b) Verify that one of the charts of your atlas satisfies the definition of a chart.

   (c) Verify that two of your charts with intersecting domains are compatible.

2. Use the following hints to prove that for $U \subset \mathbb{R}^n$ open, the map $\mathscr{C} : \mathbb{R}^n_x \to T_x U$, $v \mapsto D_v$ is a bijection, where

$$D_v(f) = \frac{d}{dt}\bigg|_{t=0} f(x+tv) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(x) v^i.$$

   (a) Show that $\mathscr{C}$ is a linear map.

   (b) Show that $\mathscr{C}(v) = 0$ (that is, $D_v(f) = 0$ for all $f \in C^\infty(M)$) implies that $v = 0 \in \mathbb{R}^n_x$.

   (c) Show that $\mathscr{C}$ is onto. That is, show that for any given $L \in T_x U$, there exists v such that $D_v = L$. To do this, consider the functions $x^i : U \to \mathbb{R}$. Set $v^i = L\left(x^i\right)$, and $v = \left(v^1, \ldots, v^n\right)$. Prove that $L = D_v$.

   Further hints: Taylor's theorem implies that for y near x,

$$f(y) = f(x) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(x)\left(y^i - x^i\right) + \sum_{i=1}^{n} g_i(y)\left(y^i - x^i\right)$$

   for some functions g_i such that $g_i(x) = 0$. (In this equation, it is helpful to view x as constant and y as variable.) Use the fact that L is linear and satisfies the Leibniz rule at x together with the formula above to prove $L(f) = D_v(f)$.

3. Consider the map

$$\begin{aligned} F : \mathbb{R}^3 \times \mathbb{R}^3 &\to \mathbb{R}^3 \\ (u, v) &\mapsto u \times v \end{aligned}$$

   where $u \times v$ is the vector cross product. Compute a matrix representative of F_*.