Math 481: Homework 1

Due Monday, February 3, 2020

1. Prove that the following sets are open:

   (a) $B_1^2((0,0))$.

   (b) $\left\{\left(x^1,x^2\right)\in\mathbb{R}^2 \mid \left(x^1\right)^2+\left(x^2\right)^2<1,\ x^2>0\right\}$.

   (c) $\left\{\left(x^1,x^2\right)\in\mathbb{R}^2 \mid x^1\neq 0\right\}$.

2. Suppose that $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ are smooth.

   (a) Show that fg is smooth.

   (b) Show that f/g is smooth on $\mathbb{R}\setminus g^{-1}(0)$.

   (c) (Bonus) Show that $f\circ g$ is smooth.

3. Consider $S^2=\left\{\left(x^1,x^2,x^3\right)\in\mathbb{R}^3 \mid \left(x^1\right)^2+\left(x^2\right)^2+\left(x^3\right)^2=1\right\}$, the subset $V_1=S^2\setminus\{(0,0,1)\}$, and the map

$$\psi_1: V_1\to\mathbb{R}^2,\quad \psi_1\left(x^1,x^2,x^3\right)=\left(\frac{x^1}{1-x^3},\frac{x^2}{1-x^3}\right)$$

   (a) Show that ψ_1 is a chart on S^2. *Hint:* Verify that

$$\psi_1^{-1}(u,v)=\left(\frac{2u}{1+u^2+v^2},\frac{2v}{1+u^2+v^2},\frac{u^2+v^2-1}{1+u^2+v^2}\right).$$

   (b) Define $V_{-1}=S^2\setminus\{(0,0,-1)\}$ and $\psi_{-1}:V_{-1}\to\mathbb{R}$ by

$$\psi_{-1}\left(x^1,x^2,x^3\right)=\left(\frac{x^1}{1+x^3},\frac{x^2}{1+x^3}\right)$$

   Show that $\mathscr{B}=\{(V_1,\psi_1),(V_{-1},\psi_{-1})\}$ is a smooth atlas on S^2.

   (c) Verify that the chart (U_1^+,ϕ_1^+) from lecture is compatible with the atlas $\mathscr{B}$.

   (d) Use $\mathscr{B}$ to show that the function $f:S^2\to\mathbb{R}$, $f(x)=\left(x^1\right)^2$ is smooth.