Math 481: Midterm 2 **Name:**

Wednesday, April 15, 2020

Instructions:

- There are 49 points possible on this exam. Take note that the problems are not weighted equally.
- Please complete the problems and upload your solutions to Box, just as you would do with a homework assignment.
- For problems that ask you to prove something, you are allowed to use in your proof any result from the lectures or the relevant sections of the textbook.
- For this take-home exam **you may refer to** your notes, the lecture videos, homework solutions, and any of resources the mentioned on the course website. **You may not** refer to online sources not mentioned on the course website (e.g. google searches), and **you may not discuss the exam with other students**.
- If you have questions (for example, needing clarification of a problem statement). Please email the instructor at `jpascale@illinois.edu`.
- This exam will be posted online no later than 7:00am CDT on Wednesday, April 15, and your solutions should be uploaded by 3:00am CDT on Saturday, April 18.

1.	2.	3.	4.	5.	Total

In the problems that follow, M refers to the surface of an ellipsoid:

$$M = \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, \frac{x^2}{2} + y^2 + \frac{z^2}{3} = 1 \right\},$$

and $\gamma : [0,2\pi] \times [0,\pi] \to M$ is a parameterization of M by

$$\gamma(u,v) = (\sqrt{2}\cos(u)\sin(v), \sin(u)\sin(v), \sqrt{3}\cos(v)), \quad (u,v) \in [0,2\pi] \times [0,\pi].$$

This parameterization is one-to-one on the interior $(0,2\pi) \times (0,\pi)$ of the domain rectangle, and so its inverse defines a coordinate system on $U = \gamma((0,2\pi) \times (0,\pi))$.
Let $i : M \to \mathbb{R}^3$ be the inclusion map $i(x,y,z) = (x,y,z)$.

1. **(12 points)** We define a Riemannian metric g on M by declaring that, for a given point $p \in M$, and given tangent vectors $X, Y \in T_pM$, we have

   $$g(p)(X,Y) = (i_*(X)) \cdot (i_*(Y))$$

   where $i_* : T_pM \to T_p\mathbb{R}^3$ is the derivative of i, and $(i_*(X)) \cdot (i_*(Y))$ denotes the ordinary dot product of the vectors $i_*(X), i_*(Y) \in T_p\mathbb{R}^3 \cong \mathbb{R}^3$.

   Write a formula for g in terms of the (u,v) coordinate system on $U \subset M$. That is, find the four coefficient functions $g_{11}, g_{12}, g_{21}, g_{22}$ such that

   $$g = g_{11}\, du \otimes du + g_{12}\, du \otimes dv + g_{21}\, dv \otimes du + g_{22}\, dv \otimes dv$$

   where each g_{ij} is a function of u and v. *Hint:* In the correct answer, each coefficient g_{ij} is combination of functions like $\sin(u), \cos(u), \sin(v), \cos(v)$. You do not need to simplify your answer.

2. **(6 points)** Set up an integral to compute length of the curve C on M parameterized by $\gamma(t, \pi/2)$, $t \in [0,2\pi]$. Your answer should be a standard single-variable integral with an explicit integrand.

3. **(3 points each)** Consider the 1-form on $\mathbb{R}^3$ given by $\alpha = dz - y\,dx$.

   (a) Compute $d\alpha$;

   (b) Compute $\alpha \wedge d\alpha$;

   (c) Compute $i^*\alpha$ in the (u,v) coordinate system;

   (d) Compute $d(i^*\alpha)$ in the (u,v) coordinate system;

   (e) Compute $i^*(\alpha \wedge d\alpha)$ in the (u,v) coordinate system.

4. **(6 points)** Let α be as in the previous problem, and let $N = M \cap \{(x,y,z) \in \mathbb{R}^3 \mid z \geq 0\}$ be the upper half of the ellipsoid. Compute $\int_N d(i^*\alpha)$; use the orientation on M that makes γ a positive parameterization.

5. Let V be a vector space. Given a vector $v \in V$, and an alternating tensor $\beta \in \Lambda^k(V)$, we can form an alternating tensor $i_v\beta \in \Lambda^{k-1}(V)$, defined by the rule

   $$(i_v\beta)(v_1, v_2, \ldots, v_{k-1}) = \beta(v, v_1, v_2, \ldots, v_{k-1})$$

   [In other words, $i_v\beta$ is like β, but the first input is always the fixed vector v, so $i_v\beta$ only depends on $k-1$ inputs.]

(a) **(5 points)** Show that if $v, w \in V$, and $\beta \in \Lambda^k(V)$, then

$$i_v(i_w\beta) = -i_w(i_v\beta),$$

where each side of the equation is an element of $\Lambda^{k-2}(V)$.

(b) **(5 points)** Assume $n = 3$ and let e_1, e_2, e_3 be a basis of V, and let $\sigma^1, \sigma^2, \sigma^3$ be the dual basis for V^*. Compute

$$i_{e_2}(\sigma^1 \wedge \sigma^2 + \sigma^2 \wedge \sigma^3)$$

The result is an element of $\Lambda^1(V) = V^*$; Write your answer in terms of the basis $\sigma^1, \sigma^2, \sigma^3$.