Lecture 41 Lagrangian spheres in the Quartic Surface.

We need a way to construct Lagrangian submanifolds in $X_0 = \{ x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0 \} \subseteq \mathbb{P}^3$

These will be as the vanishing cycles of a Lefschetz pencil in $\mathbb{P}^3$ in which X_0 appears as a member.

Quasi-Lefschetz pencils: Let $(X, \mathcal{E}_X)$ be a projective Kähler manifold. Choose a section $\sigma_{X,\infty} \in H^0(X, \mathcal{E}_X)$ such that $X_\infty = \sigma_{X,\infty}^{-1}(0)$ is a simple normal crossings divisor.
Choose another section $\sigma_{X,0} \in H^0(X, \mathcal{E}_X)$ linearly independent from $\sigma_{X,\infty}$; consider the pencil (linear series of dim 1)

$$X_z = \{ \sigma_{X,0} - z\,\sigma_{X,\infty} = 0 \} \subseteq X \qquad z \in \mathbb{P}^1$$

So $X_0 = \sigma_{X,0}^{-1}(0)$ and $X_\infty = \sigma_{X,\infty}^{-1}(0)$ as before.
The base locus $B = X_0 \cap X_\infty$ is contained in every X_z.
We require that X_0 is smooth near B and intersects each stratum of X_∞ transversally.

We call $\{X_z\}_{z \in \mathbb{P}^1}$ a quasi-Lefschetz pencil if generically X_z is nonsingular and for $z \neq \infty$, the singularities of X_z are nodes.
It is called a Lefschetz pencil if X_∞ is nonsingular as well.

This is a kind of "holomorphic Morse function".
Let $M = X \setminus X_\infty \qquad M_z = X_z \setminus B$
$\pi : M \to \mathbb{C}$, $\pi = \sigma_{X,0}/\sigma_{X,\infty}$, $\pi^{-1}(z) = M_z$.

Note X_z is projective Kähler, M_z is affine Kähler.

The condition of quasi-Lefschetz means that π is generically a submersion and its critical points are nondegenerate in the sense that the Hessian of π is nonsingular, or that $\pi \sim z_1^2 + z_2^2 + \dots + z_n^2$ near a critical point for some local holomorphic coordinates $(z_1, z_2, \dots, z_n)$ on M.

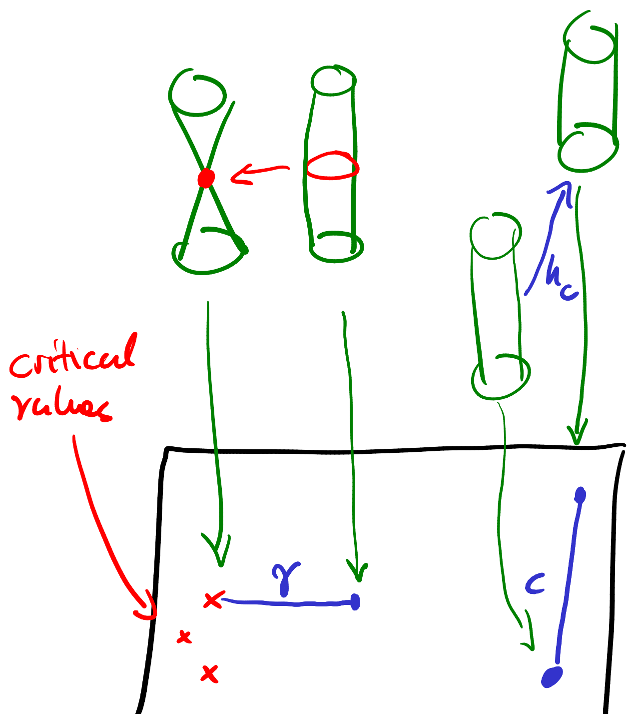
A nice thing about $\pi : M \to \mathbb{C}$ is that M has a Kähler form ω_M, and the fibers M_z are symplectic. $T_x M_z \subset T_x M$

Let $H_x = (T_x M_z)^{\perp} \subseteq T_x M$ be the distribution of subspaces that are symplectically orthogonal to the fibers of π.

Then $D\pi_x : H_x \to T_z \mathbb{C}$ is an isomorphism.

So $\{H_x\}_{x \in M}$ defines an Ehresmann connection on $\pi : M \to \mathbb{C}$

This connection is not flat, but we have parallel transport maps: Given $c : [0,1] \to \mathbb{C} \setminus (\text{critical values of } \pi)$ we get $h_c : M_{c(0)} \to M_{c(1)}$ a <u>symplectomorphism</u>.



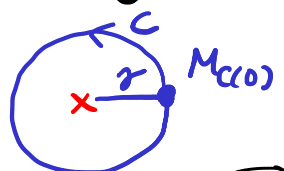
We can also consider a path $\gamma : [0,1] \to \mathbb{C}$ such that $\gamma(1) \in (\text{Critical values of } \pi)$. Then look at $V_\gamma \subseteq M_{\gamma(0)}$, the set of points that approach the critical point as we parallel transport into the critical fiber.

<u>Theorem</u> $V_\gamma \subseteq M_{\gamma(0)}$ is a Lagrangian sphere. We call it the <u>vanishing cycle</u>

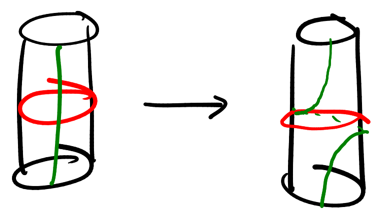
For an intuitive model, consider $M = \mathbb{C}^n$ with $\pi(x) = \sum x_i^2$
then 0 is the critical value. Take $\gamma(t) = 1-t$ then
$M_{\gamma(0)} = \{ \sum x_i^2 = 1 \}$ an affine quadric.
$M_0 = \{ \sum x_i^2 = 0 \}$ nodal quadric.

With appropriate choices $V_\gamma = \{ \sum x_i^2 = 1,\ x_i \in \mathbb{R} \} \cong S^n$

Note that if $c : [0,1] \to \mathbb{C} \setminus (\text{crit val})$ is a loop, $c(0) = c(1)$,
then $h_c \in \mathrm{Aut}(M_{c(0)})$ is a symplectic automorphism of $M_{c(0)}$
For a small loop around a critical value where the singular fiber contains a single node, this h_c is a Dehn twist (Seidel) along V_γ



Higher dimensional analog of



The first application is to construct Lagrangian spheres in the quartic surface.

$X = \mathbb{P}^3 \quad \mathcal{E}_X = \mathcal{O}_{\mathbb{P}^3}(4) \quad \sigma_{X,\infty} = x_0 x_1 x_2 x_3 \quad \sigma_{X,0} = x_0^4 + x_1^4 + x_2^4 + x_3^4$

$X_\infty = \{ x_0 x_1 x_2 x_3 = 0 \}$, union of 4 planes
$X_0 = \{ x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0 \}$, the quartic surface we care about.

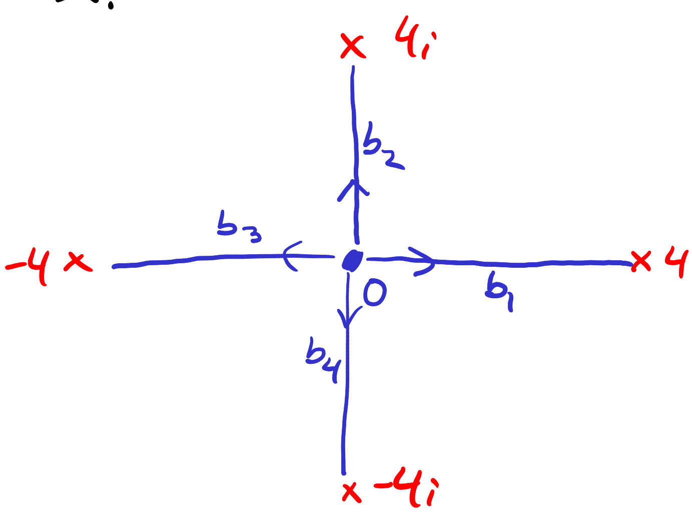
$M = X \setminus X_\infty = \{ [x_0 : x_1 : x_2 : x_3] \mid \text{all } x_i \neq 0 \}$

$$\pi : M \to \mathbb{C} \qquad \pi(x) = \frac{x_0^4 + x_1^4 + x_2^4 + x_3^4}{x_0 x_1 x_2 x_3}$$

Simple calculus and linear algebra shows that π has 64 critical points at $[1 : i^a : i^b : i^c]$ $\quad a,b,c \in \{0,1,2,3\}$

There are 4 critical values $\{\pm 4, \pm 4i\}$.

Each singular fiber has 16 nodes, So in construction of vanishing cycles, we have to say which one we are looking at. We choose to use the same path for all critical points in the same fiber.



Get 64 Lagrangian spheres in $M_0 = X_0 \setminus B$

$V_{b_1}^1, \ldots, V_{b_1}^{16}$

$V_{b_2}^1, \ldots, V_{b_2}^{16}$

$V_{b_3}^1, \ldots, V_{b_3}^{16}$

$V_{b_4}^1, \ldots, V_{b_4}^{16}$

These 64 spheres give rise to an A_∞-algebra that will match the algebraic story about Q_{64}.