Lecture 40 Affine, relative, projective Fukaya categories

A projective Kähler manifold is a compact complex manifold X with a unitary holomorphic line bundle $\mathcal{E}_X$, which is positive in the sense that if ∇_X denotes the Chern connection in $\mathcal{E}_X$, then

$$\omega_X = \frac{i}{2\pi} F_{\nabla_X} \quad \text{is a symplectic form}$$

Hence $[\omega_X] = c_1(\mathcal{E}_X) \in H^2(X;\mathbb{Z})$ is an integral class.

Let $L \subset X$ be ω_X-Lagrangian. Then $(\mathcal{E}_X, \nabla_X)|_L$ is flat. L is called rational if the holonomy of this flat connection consists of roots of unity. Equivalently, some power of $(\mathcal{E}_X, \nabla_X)|_L$ covariantly trivial. We choose a covariantly constant multisection λ_L of $(\mathcal{E}_X, \nabla_X)|_L \to L$.

The "relative" situation involves picking a holomorphic section $\sigma_{X,\infty}$ of $\mathcal{E}_X$. Then $X_\infty := \sigma_{X,\infty}^{-1}(0)$ is a divisor in X. We assume

- X_∞ is a divisor with normal crossings
- each irreducible component of X_∞ is smooth
- $\sigma_{X,\infty}$ vanishes to order 1 on each component.

We call $M = X \setminus X_\infty$ an affine Kähler manifold. Use $\dfrac{\sigma_{X,\infty}}{\|\sigma_{X,\infty}\|}$ to trivialize $\mathcal{E}_X|_M$, we can write

$\nabla_X|_M = d - 2\pi i\,\theta_M$, where $\theta_M \in \Omega^1(M)$ satisfies

$$d\theta_M = \omega_X|_M =: \omega_M. \quad \text{So } M \text{ is exact symplectic}$$

<u>Lemma</u> A Lagrangian submanifold $L \subset M$ is exact iff $\mathcal{E}_X|_L$ is trivial, and its nonzero covariantly constant sections are homotopic to $\sigma_{X,\infty}|_L$.

In particular, Exact in $M \Rightarrow$ rational in X.

Fukaya categories	Affine	Relative	Projective
Notation	$\mathcal{F}(M)$	$\mathcal{F}(X, X_\infty)$	$\mathcal{F}(X)$
coefficients	$\mathbb{C}$	$\Lambda_{\mathbb{N}}$	$\Lambda_{\mathbb{Q}}$
restriction on Lagrangians	exact	exact $\subset M$	rational

<u>Affine</u> $\mathcal{F}(M)$ is the category formed by exact Lagrangian branes

$(L, \tilde{\alpha}_L, \delta_L)$ L oriented, closed, exact
$\tilde{\alpha}_L$ grading
δ_L spin structure.

The category is defined in the way we know

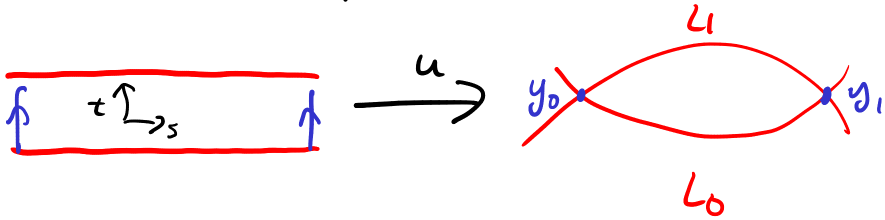
<u>Projective</u> Objects in $\mathcal{F}(X)$ are <u>rational Lagrangian branes</u>

$$(L, \tilde{\alpha}_L, \delta_L, \lambda_L, J_L)$$

- L oriented closed Lagrangian submanifolds of X
- $\tilde{\alpha}_L$, δ_L grading + spin structure
- λ_L is a covariantly constant multisection of $\mathcal{E}_X|_L$
- J_L is a ω_X-compatible almost complex structure which is "regular" for L (for technical reasons we must include this in the definition of the objects.)

This category is defined over $\Lambda_{\mathbb{Q}} = \overline{\mathbb{C}((q))}$, and we must say how the q-powers come in.

For the differential $\mu^1 : HF(L_0, L_1) \circlearrowleft$ we count strips



Each strip has an energy $E(u) = \int |\partial_s u|^2 \, ds\, dt > 0$

Our rationality assumption implies that $E(u) \in \mathbb{Q}$
The contribution of u to $\mu^1(y_1)$ is $\pm q^{E(u)} y_0$

Although there may be infinitely many u's contributing, Gromov compactness implies that only finitely many terms have $E(u) < C$. Also $E(u)$ that appear in a given calculation can be put over a common denominator, so the operation is defined over $\Lambda_{\mathbb{Q}} = \overline{\mathbb{C}((q))}$.

For the A_∞-operations μ^d, we do something similar

$$u : S \to X \qquad (du - Y)^{0,1} = 0 \qquad Y = \text{sgrad } K_S$$

$$E(u) = \int_S |du - Y|^2 \, dvol$$

$$E(u) - \int_S u^*\left(dK_S + \tfrac{1}{2}\{K_S, K_S\}\right) \in \mathbb{Q}$$

weight u by q to this power.

<u>Remark</u>: different choices of λ_L give isomorphic objects. The isomorphism is multiplication by q^m for some $m \in \mathbb{Q}$.

<u>Relative</u> $\mathcal{F}(X, X_\infty)$ has objects similar to $\mathcal{F}(M)$ and composition similar to $\mathcal{F}(X)$.

Objects $(L, \tilde{\alpha}_L, \mathfrak{s}_L, J_L)$ L <u>exact</u> Lagrangian in M

The chain groups are $\Lambda_{\mathbb{N}}$-modules, and the contribution of $u : S \to M$ to an operation is weighted by $q^{u \cdot X_\infty}$

the statement $u \cdot X_\infty \in \mathbb{N}$ is a version of positivity of intersection for curves in a surface.

<u>Proposition</u>: Consider $\mathbb{C} = \Lambda_{\mathbb{N}} / (q)$ as a $\Lambda_{\mathbb{N}}$ module

then $\mathcal{F}(X, X_\infty) \otimes_{\Lambda_{\mathbb{N}}} \mathbb{C} \cong \mathcal{F}(M)$

<u>Proof</u> setting $q=0$ ignores all u such that $u \cdot X_\infty > 0$

the maps such that $u \cdot X_\infty = 0$ are exactly those that contribute to composition in $\mathcal{F}(M)$.

<u>Proposition</u> Consider $\Lambda_{\mathbb{R}}$ as a $\Lambda_{\mathbb{N}}$-module in the obvious way.

Then $\mathcal{F}(X, X_\infty) \otimes_{\Lambda_{\mathbb{N}}} \Lambda_{\mathbb{R}}$ is equivalent to a full subcategory of $\mathcal{F}(X)$

<u>Proof</u> The q weights are <u>not</u> exactly the same, but they can be matched by multiplying a basis element $x \in CF(L_0, L_1)$ by $q^{A(x)}$, where $A(x)$ is the action of x.

We can think of $\mathcal{F}(X, X_\infty)$ as a 1-parameter family of categories whose general fiber embeds in $\mathcal{F}(X)$ and whose central fiber is $\mathcal{F}(M)$.