Lecture 39   Results for mirror of quartic surface

Q_4 = total morphism algebra of C_4 where

$$\mathrm{Hom}_{C_4}(X_j, X_k) = \begin{cases} \Lambda^{k-j} V & j<k \\ \Lambda^0 V \oplus (\Lambda^4 V)[2] & j=k \\ \Lambda^{k-j+4} V & j>k \end{cases}$$

it's an algebra over $R_4 = \mathbb{C}\times\mathbb{C}\times\mathbb{C}\times\mathbb{C}$

$\Gamma_4 \subset SL(V)$ gen by $i\,\mathrm{Id}_V$, $H \subset SL(V)$ max torus

$T = H/\Gamma_4$ acts on Q_4.

$HH^4(Q_4, Q_4[-2])^T$ is rank 1

And there is a unique non formal T-invariant A_∞-structure $\mathcal{Q}_4$ on Q_4 up to R_4-linear T-equivariant A_∞ isomorphism.

Let $\Gamma_{64} \subset H \subset SL(V)$ be the group of diagonal matrices that have order 4, and let $\Gamma_{16} = \Gamma_{64}/\Gamma_4$

then Γ_{16} acts on Q_4, and on $\mathcal{Q}_4$, and we form

$$Q_{64} = Q_4 \rtimes \Gamma_{16} \quad \text{(graded algebras)}$$

$$\mathcal{Q}_{64} = \mathcal{Q}_4 \rtimes \Gamma_{16} \quad \text{(nonformal } A_\infty\text{-algebras)}$$

We then need a result about $HH^2(\mathcal{Q}_{64}, \mathcal{Q}_{64})^{\leq 0}$

it turns out the space is 7-dimensional, but we can cut it down by considering further symmetry.

Let $\mathcal{U}_4 \in GL(V)$ be $\begin{pmatrix} 0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0 \end{pmatrix}$ cyclic perm of coordinates

$\mathcal{U}_4$ lifts to a $\mathbb{Z}/4$ action on Q_4, get $\mathcal{U}_{64}$ generating a $\mathbb{Z}/4$ action on Q_{64}

It is possible to make $\mathcal{Q}_4$ invariant under $\mathcal{U}_4$, and hence make $\mathcal{Q}_{64}$ invariant under $\mathcal{U}_{64}$. We do this henceforth.

The next result is $HH^2(\mathcal{Q}_{64}, \mathcal{Q}_{64})^{SO, \mathbb{Z}/4} \cong \mathbb{C} \cdot (y_0^4 + y_1^4 + y_2^4 + y_3^4)$
so has rank 1 [$\mathrm{Sym}^4(V^\vee)$ appears as a summand in $HH^2(\mathcal{Q}_{64}, \mathcal{Q}_{64})^{SO}$]

Now applying our one-parameter deformation theory results,
A $\mathcal{U}_{64}$-invariant one-parameter deformation $\mathcal{Q}_{64,q}$ with nontrivial deformation class is unique up to gauge and reparametrization by $\mathrm{End}(\Lambda_N)^\times$
(If it exists – which it does as we shall see).

Now consider $Z_q^* = \widetilde{Y_q^* / \Gamma_{16}}$ $\quad Y_q^* = \{ y_0y_1y_2y_3 + q(y_0^4 + y_1^4 + y_2^4 + y_3^4) = 0 \}$ in $\mathbb{P}^3_{\Lambda_\mathbb{Q}}$

<u>Theorem</u> $D^b\mathrm{Coh}(Z_q^*) \cong D^b\mathrm{Coh}_{\Gamma_{16}}(Y_q^*) \cong H^0\left((\pi^* \mathcal{Q}_{64,q} \otimes_{\Lambda_N} \Lambda_\mathbb{Q})^{perf} \right)$

The first equivalence is Kapranov-Vasserot.
For the second, we argue as follows:
Let $F_k = \Omega^{4-k}(4-k)[4-k]$ on $\mathbb{P}^3$ Beilinson basis.
Let $E_{0,k} = i_0^*$ $\quad i_0 : Y_0 \to \mathbb{P}^3 \quad Y_0 = \{ y_0y_1y_2y_3 = 0 \}$

Let S_4 = total morphism algebra of $\{E_{0,4}\}$
then $S_4 \cong Q_4$. Let $\mathcal{S}_4$ be A_∞-structure from $\mathrm{Coh}(Y_0)$
<u>Lemma</u> The A_∞-structure on $\mathcal{S}_4$ is not formal.
It is T-invariant, so $\mathcal{S}_4 \cong \mathcal{Q}_4$
Now consider $D^b\mathrm{Coh}_{\Gamma_{16}}(Y_0)$ each $E_{0,k}$ has 16 equivariant structures, so we get a total of 64 equivariant objects. Let $\mathcal{S}_{64}$ be the total morphism A_∞-algebra of these objects.
Then $\mathcal{S}_{64} \cong \mathcal{Q}_{64}$.

Now consider $Y_q = \{ y_0 y_1 y_2 y_3 + q(y_0^4 + y_1^4 + y_2^4 + y_3^4) = 0 \}$ over Λ_N

This has a Γ_{16} action, and $E^{\chi}_{0,k}$ all extend to $E^{\chi}_{q,k}$

$\mathcal{S}_{64,q}$ be the total endomorphism algebra from $\mathrm{Coh}_{\Gamma_{16}}(Y_q)$

<u>Lemma</u> $\mathcal{S}_{64,q}$ has non trivial deformation class in $HH^2(\mathcal{S}_{64}, \mathcal{S}_{64})$

We can make $\mathcal{S}_{64,q}$ $\mathcal{U}_{64}$-invariant, so

$$\mathcal{S}_{64,q} \cong \psi^* \mathcal{Q}_{64,q} \quad \text{for some } \psi \in \mathrm{End}(\Lambda_N)^{\times}$$

Let $Y_q^* = Y_q \times_{\Lambda_N} \Lambda_Q$ let $E^{\chi,*}_{q,k}$ be corresponding sheaves.

<u>Lemma</u> The 64 objects $E^{\chi,*}_{q,k}$ split generate $D^b\mathrm{Coh}_{\Gamma_{16}}(Y_q^*)$

Thus

$$\begin{array}{ccc} D^b\mathrm{Coh}_{\Gamma_{16}}(Y_q^*) & \cong & H^0\left(\left(\mathcal{S}_{64,q} \otimes_{\Lambda_N} \Lambda_Q\right)^{perf}\right) \\ \wr\!\!/\!\!/ & & \wr\!\!\mid\!\!\mid \\ D^b\mathrm{Coh}(Z_q^*) & & H^0\left(\left(\psi^* \mathcal{Q}_{64,q} \otimes_{\Lambda_N} \Lambda_Q\right)^{perf}\right) \end{array}$$

$\blacksquare$