Lecture 37 A_∞-structures on Q_4, Deformation theory.

Recall the algebra Q_4 which is the total morphism algebra of the category/$\mathbb{C}$

$$\mathrm{Hom}(X_i, X_j) = \begin{cases} \Lambda^{k-j} V & j<k \\ \Lambda^0 V \oplus (\Lambda^4 V)[2] & j=k \\ (\Lambda^{k-j+4} V)[2] & j>k \end{cases}$$

where V is a 4-dim $\mathbb{C}$-vectorspace.

Another way to present this algebra is as $\Lambda V \rtimes \Gamma_4$ where $\Gamma_4 \subset SL(V)$ is the center, generated by $\gamma = i \cdot \mathrm{Id}_V$

This is an algebra over $\mathbb{C}\Gamma_4 \cong R_4 := \mathbb{C}\times\mathbb{C}\times\mathbb{C}\times\mathbb{C}$ (semisimple ring w/ 4 idempotents)

$\mathbb{C}\Gamma_4$ has a basis of idempotents given by

$$e_j = \frac{1}{4}\left(e + i^{-j}\gamma + i^{-2j}\gamma^2 + i^{-3j}\gamma^3\right)$$

The possible A_∞-structures extending the given algebra structure of Q_4 are controlled by $HH^d(Q_4, Q_4[2-d])$. We find (Seidel, HMS for QS., (10.7)).

$$HH^d(Q_4, Q_4[2-d]) = \begin{cases} \mathrm{Sym}^4(V^\vee) & d=4 \\ 0 & \text{other } d=3,\ d\geq 5 \end{cases}$$

So the A_∞-structure is again governed by a single obstruction class. Of course $\dim \mathrm{Sym}^4(V^\vee) = \binom{7}{4} = 35$ so there are a lot of possibilities. However, we can reduce the dimension by considering A_∞-structures compatible with extra symmetries.

$GL(V)$ acts on Q_4, and so does $I_4 = (\mathbb{C}^*)^4/\mathbb{C}^*$, where $\alpha = [\alpha_1, \alpha_2, \alpha_3, \alpha_4]$ acts on $e_k Q_4 e_j$ by multiplication by α_k / α_j. These actions commute and we find

$$\mathrm{Aut}_{R_4}(Q_4) \cong I_4 \times_{\Gamma_4} GL(V)$$

Next let $H \subseteq SL(V)$ be the maximal torus of diagonal matrices, and let $T = H/\Gamma_4$.

The map $T \to GL(V)/\Gamma_4$ lifts to a map $T \to \mathrm{Aut}_{R_4}(Q_4)$ and we can look for T-invariant A_∞-structures.

The computation of Hochschild cohomology respects the Taction, so

$$HH^4(Q_4, Q_4[-2])^T \cong \mathbb{C} \cdot (y_0 y_1 y_2 y_3)$$

So there is only 1-dimensional space of T-invariant deformations. Suppose $\mathcal{Q}_4$ is a non-formal T-invariant A_∞-structure on Q_4. Then

$\mathcal{Q}_4$ is unique upto R_4-linear T-equivariant A_∞-isomorphism.

$\mathcal{Q}_4$ does appear in algebraic geometry. Consider

$Y_0 = \{y_0 y_1 y_2 y_3 = 0\} \subseteq \mathbb{P}(V)$. It is reduced by has several components. $i_0 : Y_0 \to \mathbb{P}(V)$

We have the Beilinson generators $F_k = \Omega^{4-k}(4-k)[4-k]$ on $\mathbb{P}(V)$ we denote $E_{0,k} = i_0^* F_k$. There is an A_∞-category $\mathrm{Perf}(Y_0)$ where objects are perfect complexes on Y_0 then

$$\mathcal{Q}_4^{\mathrm{perf}} \cong \mathrm{Perf}(Y_0)$$
$$X_k \to E_{0,k}$$

Next we need to turn on the q-parameter in $\mathbb{C}[[q]] = \Lambda_{\mathbb{N}} \subseteq \Lambda_{\mathbb{Q}}$, and divide by the Γ_{16}-action.

This gives rise to a slightly different deformation problem. We start with an A_∞-algebra $\mathcal{A}/\mathbb{C}$. Then we extend scalars to $\mathcal{A}_q = \mathcal{A} \underset{\mathbb{C}}{\otimes} \Lambda_{\mathbb{N}}$. We look for A_∞ structures on $\mathcal{A}_q$ that reduce to the given one mod q. So

$$\mu^d_{\mathcal{A}_q} = \mu^d_{\mathcal{A}} + q\,\mu^d_{\mathcal{A}_{q,1}} + q^2 \mu^d_{\mathcal{A}_{q,2}} + \dots = \mu^d_{\mathcal{A}} + O(q)$$

We call $\mathcal{A}_q$ a <u>one-parameter deformation</u> of $\mathcal{A}$.
Two one-parameter deformations are gauge-equivalent if there is a $\Lambda_{\mathbb{N}}$-linear A_∞-isomorphism $\mathcal{G}_q$ of the form

$$\mathcal{G}_q^d = \begin{cases} id + O(q) & d = 1 \\ O(q) & d > 1 \end{cases}$$

Let $\mathcal{U}_q(\mathcal{A}) = \{\text{one-parameter defs } \mathcal{A}_q\}$
$\mathcal{G}_q(\mathcal{A}) = \{\text{gauge transformations as above}\}$

We wish to study $\mathcal{U}_q(\mathcal{A}) / \mathcal{G}_q(\mathcal{A})$

Let $\mathrm{End}(\Lambda_{\mathbb{N}})$ be the semigroup of q-adically continuous ring maps $\Lambda_{\mathbb{N}} \to \Lambda_{\mathbb{N}}$ each such map is of the form $q \mapsto \psi(q)$

This acts on $\mathcal{U}_q(\mathcal{A})$ by $\mathcal{A}_q \mapsto \psi^* \mathcal{A}_q$ twisting the $\Lambda_{\mathbb{N}}$-linear structure.

This deformation problem is governed by the Hochschild cohomology of the A_∞-algebra $\mathcal{A}$. This is slightly different from the HH^* of a graded algebra, since the differential now involves higher μ^d. Concretely, we take

$$CC^d(\mathcal{A},\mathcal{A}) = \prod_{s+t=d} Hom\left(\mathcal{A}^{\otimes s}, \mathcal{A}[t]\right)$$

$$(\sigma \circ \tau)^d(a_d,\dots,a_1) = \sum_{i,j} (-1)^{\#} \sigma^{d-j+1}(a_d,\dots,\tau^j(a_{i+j},\dots,a_{i+1}),\dots,a_1)$$

$$\# = (|\tau|-1)(|a_1|+\dots+|a_i|-1)$$

$$[\sigma,\tau] = \sigma\circ\tau - (-1)^{(|\sigma|-1)(|\tau|-1)}\tau\circ\sigma \qquad \text{Gerstenhaber bracket}$$

$\mu^*_{\mathcal{A}}$ defines a cochain in $CC^2(\mathcal{A},\mathcal{A})$, and the differential is

$$\delta\tau = [\mu^*_{\mathcal{A}},\tau]$$

This $(CC^*(\mathcal{A},\mathcal{A}), [\,,\,], \delta)$ is a dg-Lie algebra, and we can use deformation theory à la Deligne to understand 1-parameter deformations.