Lecture 36 Start the quartic surface.

Let $X_0 \subseteq \mathbb{CP}^3$ be a quartic surface, equipped with its standard symplectic structure (restriction of the Fubini-Study Kähler form on $\mathbb{CP}^3$).
Up to symplectomorphism, X_0 does not depend on the equation that defines it so we can take, for instance $X_0 = \{x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\}$

Following Seidel, we will use a version of the Fukaya category that is defined over

$$\Lambda_{\mathbb{Q}} = \left\{ f(q) = \sum_m a_m q^m \;\middle|\; \begin{array}{l} a_m \in \mathbb{C},\ m \text{ runs over } \\ (\tfrac{1}{d}\mathbb{Z}) \subseteq \mathbb{Q} \text{ for some } d \text{ depending on } f, \\ a_m = 0 \text{ for } m \ll 0 \end{array} \right\}$$

[Slightly different from $R = \{\sum a_k t^{m_k} \mid \lim m_k \to \infty\}$
but $\Lambda_{\mathbb{Q}} \to R \;:\; q \mapsto t$]

The category $Fuk(X_0)$ is defined over $\Lambda_{\mathbb{Q}}$, provided we restrict to "rational" Lagrangians

The mirror variety to X_0 is expected, from physics arguments, to be Z_q^*, constructed as follows.

Take $Y_q^* = \{y_0 y_1 y_2 y_3 + q(y_0^4 + y_1^4 + y_2^4 + y_3^4) = 0\} \subseteq \mathbb{P}^3_{\Lambda_{\mathbb{Q}}}$

Set $\Gamma_{16} = \{[\mathrm{diag}(\alpha_0, \alpha_1, \alpha_2, \alpha_3)] \mid \alpha_k^4 = 1,\ \alpha_0\alpha_1\alpha_2\alpha_3 = 1\} \subseteq PSL(4, \mathbb{C})$

($\Gamma_{16} \cong \mathbb{Z}_4 \times \mathbb{Z}_4$) this acts on Y_q^*

Set $Z_q^* = \widetilde{Y_q^*/\Gamma_{16}}$, where $\sim$ denotes the minimal resolution of the (singular) quotient surface.

Seidel's theorem for the quartic surface is then

$$\mathrm{Fuk}(X_0)^{perf} \cong \hat{\psi}^* D^b\mathrm{Coh}(Z_q^*)$$

where $\hat{\psi}$ is an automorphism of $\Lambda_{\mathbb{Q}}/\mathbb{C}$ that is used to twist the $\Lambda_{\mathbb{Q}}$-linear structure. ψ is known as a "mirror map."

The proof has steps that are somewhat analogous to what we went through for the torus, but they are more complicated. On the symplectic side, there are several new ingredients to handle this higher dimensional case.

One thing common to both is a detailed study of the deformation theory of a quiver algebra Q_{64} with 64 vertices.
Let us approach this from the algebraic side

$$Z_q^* = \widetilde{Y_q^*/\Gamma_{16}} \qquad Y_q^* \subseteq \mathbb{P}^3_{\Lambda_{\mathbb{Q}}}$$

Beilinson quiver for $\mathbb{P}^3$: Let V be a 4-dim vector space, $\mathbb{P}^3 = \mathbb{P}(V)$
This comes from considering the objects $F_k = \Omega^{4-k}(4-k)[4-k]$ $k=1,2,3,4$
The resulting quiver has 4 vertices; Call it $\vec{C_4}$

$$\vec{C_4} \qquad \begin{tikzcd} x_1 \arrow[r, "V"] \arrow[rr, bend right, "\Lambda^2 V"'] \arrow[rrr, bend right=60, "\Lambda^3 V"'] & x_2 \arrow[r, "V"] \arrow[rr, bend right, "\Lambda^2 V"'] & x_3 \arrow[r, "V"] & x_4 \end{tikzcd}$$

$$\mathrm{Hom}_{\vec{C_4}}(x_j, x_k) = \begin{cases} \Lambda^{k-j} V & j \leq k \\ 0 & \text{otherwise} \end{cases}$$

It has the natural grading of the exterior algebra and composition comes from wedge product.

Beilinson's theorem may be stated as $D^b\mathrm{Coh}(\mathbb{P}^3) \cong (\vec{C_4})^{\mathrm{perf}}$

Next we pass to a quartic hypersurface Y_q^*.
We can restrict F_k to Y_q^*, and we get a quiver with the same vertices but more arrows. This is because Serre duality + $\omega_{Y_q^*} \cong$ triv implies that Hom spaces on Y_q^* are self-dual up to shift by $\dim Y_q^* = 2$

The result is a category C_4 with 4 objects X_1, X_2, X_3, X_4

$$\mathrm{Hom}_{C_4}(X_j, X_k) = \begin{cases} \Lambda^{k-j} V & j < k \\ \Lambda^0 V \oplus (\Lambda^4 V)[2] & j = k \\ (\Lambda^{k-j+4} V)[2] & j > k \end{cases}$$

We denote by Q_4 the total morphism algebra of C_4.

Y_q^* corresponds to some A_∞-structure on this algebra, that we must figure out.

Next, we must pass from Y_q^* to $Z_q^* \cong \widetilde{Y_q^*/\Gamma_{16}}$.
It turns out that in the realm of derived categories, this is <u>easier</u> than you might think

A Theorem of Kapranov-Vasserot implies

$$D^b\mathrm{Coh}(Z_q^*) \cong D^b\mathrm{Coh}_{\Gamma_{16}}(Y_q^*), \text{ where}$$

the RHS is Γ_{16}-equivariant coherent sheaves.

In terms of quiver algebras, the equivariant structure corresponds to taking semidirect product.

Now $GL(V)$ naturally acts on Q_4, and the action of $\Gamma_{16} \subseteq PSL(V)$ can be lifted to an action of Γ_{16} on Q_4.

We then consider $Q_{64} = Q_4 \rtimes \Gamma_{16}$. A certain A_∞-structure on this algebra recovers $D^b Coh(Z_q^*)$.

For a $\mathbb{C}$-algebra A and a group Γ and a homomorphism $\rho: \Gamma \to Aut(A)$, the semidirect product is

$$A \rtimes_\rho \Gamma = A \otimes \mathbb{C}\Gamma \quad \text{with multiplication}$$

$$(a \otimes \gamma)(a' \otimes \gamma') = a\,\rho(\gamma)(a') \otimes \gamma\gamma'$$

It is also called "skew group ring" or "smash product"]

The algebra Q_{64} can be regarded as a category with 64 objects. We need to understand what A_∞-structures on Q_{64} can look like.

We also need to find 64 objects in $Fuk(X_0)$ that correspond $\longrightarrow$ some interesting symplectic topology here!