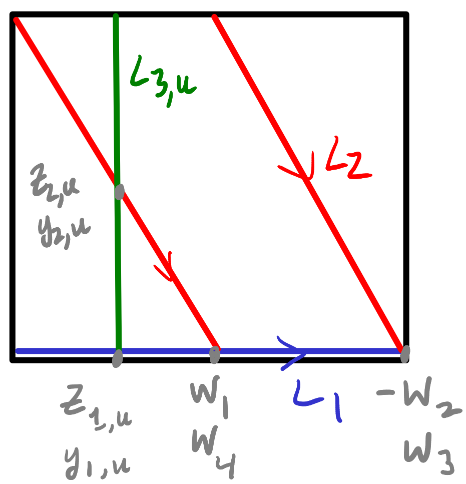
Lecture 35 Conclusion of HMS for the two-torus

Recall picture:



Let $u \notin \{\pm t^{\frac{1}{2}k} \mid k \in \mathbb{Z}\}$, so that $HF^*(L_{3,u}, L_{3,u^{-1}}) = 0$.
Consider the triangle

$$\begin{tikzcd} L_1 \arrow[rr, "v"] & & L_2 \arrow[dl, "{([z_{2,u}],[z_{2,u^{-1}}])}"] \\ & L_{3,u} \oplus L_{3,u^{-1}} \arrow[ul, "{[1]}", "{([y_{1,u}],-[y_{1,u^{-1}}])}"'] & \end{tikzcd}$$

$$v = \frac{\Theta_{2,2}(u)[w_1] + \Theta_{2,1}(u)[w_2]}{\Theta_{4,3}'(1)\left(\Theta_{4,1}(u) - \Theta_{4,3}(u)\right)}$$

The computations of the compositions in $H^0(\mathrm{Fuk}(T)^{TW})$ show that the composition of any two consecutive morphisms vanishes.

Lemma: This is an exact triangle in $H^0(\mathrm{Fuk}(T)^{TW})$

To prove this we need

$$\begin{array}{c} CF^0(L_2, L_{3,u}) \otimes CF^0(L_1, L_2) \otimes CF^1(L_{3,u}, L_1) \\ \downarrow \mu^3 \\ CF^0(L_{3,u}, L_{3,u}) \end{array}$$

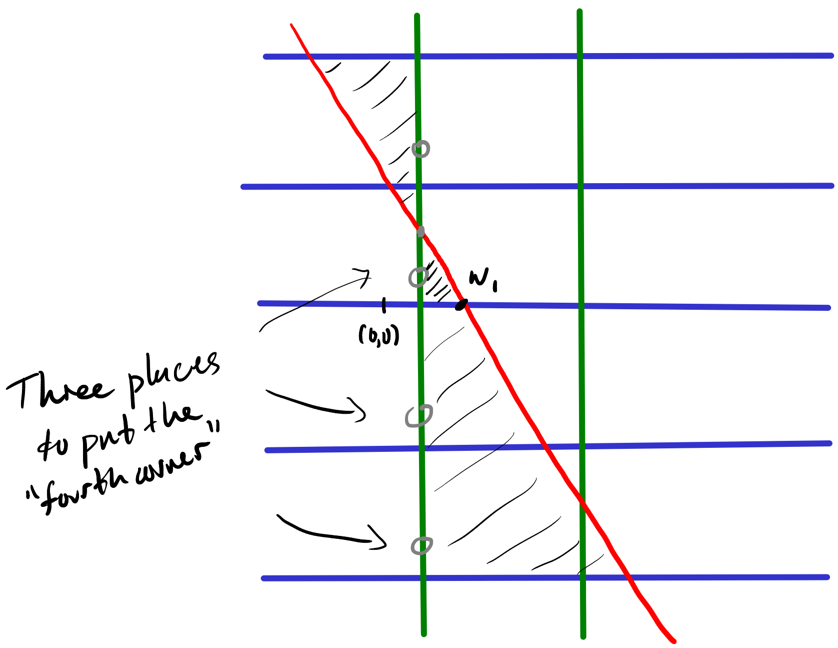
$$\mu^3(z_{2,u}, w_1, y_{1,u}) = \left(-u\Theta_{2,1}'(u) + b\Theta_{2,1}(u)\right) e_{3,u}$$

$$\mu^3(z_{2,u}, w_2, y_{1,u}) = \left(u\Theta_{2,2}'(u) - b\Theta_{2,2}(u)\right) e_{3,u}$$

We are assuming $CF^*(L_{3,u}, L_{3,u})$ is minimal $(\mu^1 \equiv 0)$ and $e_{3,u}$ is a cochain representing the identity morphism.

The constant $b \in \mathbb{R}$ represents a legitimate ambiguity: it depends on how $CF^*(L_{3,u}, L_{3,u})$ is set up.

We can arrange $b = 0$ if we use a Morse-Bott model where $CF^*(L_{3,u}, L_{3,u})$ is generated by critical points of a perfect morse function on $L_{3,u}$. Then μ^3 is determined by counting triangles with sides on $(L_{3,u}, L_2, L_1)$ and an additional marked point on the boundary that goes through the minimum.



coeff of $e_{3,u}$ in $\mu^3(z_{2,u}, w_1, y_{1,u})$

$$= \cdots - \hbar^{1/4} u + \hbar^{1/4} u^{-1} + 3\hbar^{9/4} u^{-3} + \cdots$$
$$= -u\,\theta_{2,1}'(u)$$

Proof of lemma: We know that the composition of any two maps in the triangle is zero on chain level. Denote

$$C_v = \mathrm{Cone}(v : L_1 \to L_2) \qquad v = \frac{\theta_{2,2}(u)[w_1] + \theta_{2,1}(u)[w_2]}{\theta_{4,3}'(1)\left(\theta_{4,1}(u) - \theta_{4,3}(u)\right)}$$

Then $(0, z_{2,u}) \in hom^0_{Fuk(T)^{TW}}(C_V, L_{3,u}) = CF^{-1}(L_1, L_{3,u}) \oplus CF^0(L_2, L_{3,u})$

$(y_{1,u}, 0) \in hom^0_{Fuk(T)^{TW}}(L_{3,u}, C_V) = CF^1(L_{3,u}, L_1) \oplus CF^0(L_{3,u}, L_2)$

are cocycles. Also

$$\mu^2_{Fuk(T)^{TW}}((0,z_{2,u}),(y_{1,u},0)) = \mu^3_{Fuk(T)}(z_{2,u}, V, y_{1,u}) = e_{3,u}$$

where we use the μ^3 computation above together with the theta function identity

$$t\left(\Theta'_{2,2}(t)\Theta_{2,1}(t) - \Theta'_{2,1}(t)\Theta_{2,2}(t)\right) = \Theta'_{4,3}(1)\left(\Theta_{4,1}(t) - \Theta_{4,3}(t)\right)$$

The analogous properties hold for $(0, z_{2,u^{-1}})$ $(-y_{1,u^{-1}})$.
This shows that, in $Fuk(T)^{perf}$, there is a splitting

$$C_V \cong L_{3,u} \oplus L_{3,u^{-1}} \oplus N$$

By comparison of the sizes of endomorphism rings, we find $N = 0$.
Thus $C_V \cong L_{3,u} \oplus L_{3,u^{-1}}$. This isomorphism is compatible with the exact triangle.

(Remark: If $u \in \{\pm h^{\frac{1}{2}k} \mid k \in \mathbb{Z}\}$, then $L_{3,u} \cong L_{3,u^{-1}}$, so we don't really get two summands from this argument.)

Now we know C_V splits as a direct sum if
$V = \mathbb{C}(\Theta_{2,2}(u)[W_1] + \Theta_{2,1}(u)[W_2])$ $\quad u \notin \{\pm h^{\frac{1}{2}k} \mid k \in \mathbb{Z}\}$

so the quartic $p \in Sym^4(V^\vee)$ such that $\bigoplus_{i,j=1}^{2} HF^*(L_i, L_j) \underset{\Lambda_\infty}{\cong} Q_p$

must not vanish at these points.

Thus, the vanishing locus of p in $\mathbb{P}^1(V)$ has support contained in the set

$$\{[\theta_{2,2}(1):\theta_{2,1}(1)],\ [\theta_{2,2}(-1):\theta_{2,1}(-1)],$$
$$[\theta_{2,2}(t_i^{1/2}):\theta_{2,1}(t_i^{1/2})],\ [\theta_{2,2}(-t_i^{1/2}):\theta_{2,1}(-t_i^{1/2})]\}$$

There are symmetries in the problem that show that the order of vanishing must be the same at all 4 points. (Seidel)
This implies that p is a multiple of

$$p(v_1,v_2) = c\left(\theta_{2,2}(t_i^{1/2})^2 v_2^2 - \theta_{2,1}(t_i^{1/2})^2 v_1^2\right)\left(\theta_{2,2}(1)^2 v_2^2 - \theta_{2,1}(1)^2 v_1^2\right)$$

The constant doesn't matter up to quasiisomorphism, though it can be calculated.

Proposition Q_p^{perf} is quasi equivalent to $\mathrm{Fuk}(T)^{perf}$ (where p is as above)

From what we have calculated, we get a fully faithful A_∞-functor $Q_p^{perf} \to \mathrm{Fuk}(T)^{perf}$

$$1 \mapsto L_1$$
$$2 \mapsto L_2$$

We will be done if we show L_1 and L_2 split-generate $\mathrm{Fuk}(T)^{perf}$.
Clearly, L_1 and L_2 split generate $L_{3,u}$ for generic u, since $C_v = L_{3,u} \oplus L_{3,u^{-1}}$.
Then we may appeal to a result of Seidel [FCPLT, Cor. 5.8] to show that L_1 and $L_{3,u}$ split generate $\mathrm{Fuk}(T)^{perf}$. ∎

On the other hand $Q_p^{perf} \cong D^b(\mathrm{Coh}\, Y_p)$ so HMS is proved!