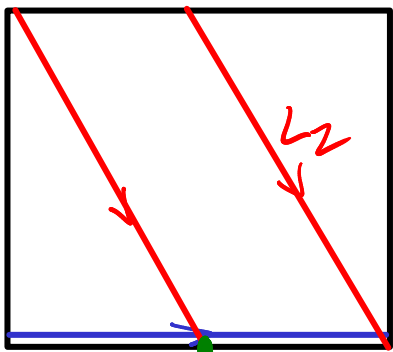


# Lecture 34 Twisted complexes on the two torus

Recall:  $T = \mathbb{R}^2 / \mathbb{Z}^2$  coords  $(p, q)$   
 $L_1 = \{q=0\}$   $L_2 = \{q=-2p\}$



We will now consider other objects of the form

$$L = \{p=m_0\}$$

$$\begin{array}{l} L_1 \\ w_1 \\ w_4 \end{array} \quad \begin{array}{l} -w_2 \in CF^0(L_1, L_2) \\ w_3 \in CF^1(L_2, L_1) \end{array}$$

More specifically, recall  $R = \left\{ u = \sum c_k t^{m_k} \mid c_k \in \mathbb{C}, m_k \in \mathbb{R}, \lim_{k \rightarrow \infty} m_k = \infty \right\}$

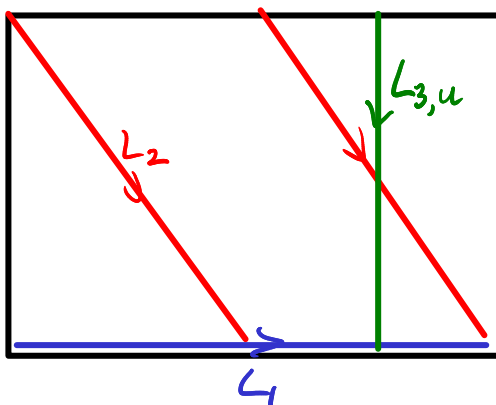
Let  $u \in R^\times$ . There is a unique factorization  $u = t^{m_0} a$   
 with  $m_0 \in \mathbb{R}$  and  $a = a_0 + a_1 t^{m_1} + a_2 t^{m_2} + \dots$   $m_i \uparrow 0$   
 $a_0 \neq 0$

$m_0 = \text{val}(u)$  is called the valuation of  $u$ .

Given  $u \in R^\times$ , we define an object  $L_{3,u}$  in  $\text{Fuk}(T)$ :

Write  $u = t^{m_0} a$ ,  $m_0 = \text{val}(u)$ . The underlying curve is

$$L_{3,u} = \{p=m_0\}$$



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We give it a grading such that  $HF^*(L_1, L_3, u)$  and  $HF^*(L_2, L_3, u)$  are concentrated in degree 0.

We also equip  $L_{3,u}$  with a rank 1  $R$ -local system  $\xi_u \rightarrow L_{3,u} \times \mathbb{R}$  such that the holonomy in the positive  $q$ -direction is  $a \in R$ .

[This is something we can always do in Floer theory, it has the effect that all compositions now carry an extra factor that measures the holonomy of the local systems along the boundaries of the disk.]

Observe that since  $p$  is taken mod 1,  $u$  and  $tu$  give the same object. So effectively  $u \in R^x / \pi \mathbb{Z}$ .

Morphism spaces

$$CF^0(L_1, L_3, u) \cong (\xi_u)_{(m_0, 0)}$$

$$CF^0(L_2, L_3, u) \cong (\xi_u)_{(m_0, -2m_0)}$$

We identify these with  $R$  by the following conventions.

Pick an arbitrary isomorphism

$$(\xi_u)_{(m_0, 0)} \cong R$$

Choose  $(\xi_u)_{(m_0, -2m_0)} \cong R$  in such a way that parallel transport along  $\gamma(t) = (m_0, -2m_0 t)$   $0 \leq t \leq 1$  is multiplication by  $\pi^{m_0^2}$ .

Denote generators corresponding to  $1 \in R$  under these isos by

$$z_{1,u} \in CF^0(L_1, L_3, u) \quad z_{2,u} \in CF^0(L_2, L_3, u)$$

Then we may compute the product

$$HF^0(L_2, L_{3,u}) \otimes HF^0(L_1, L_2) \rightarrow HF^0(L_1, L_{3,u})$$

by counting triangles weighted by area on the holonomy, similar to lecture 28. The result is

$$[z_{2,u}] \cdot [w_1] = \Theta_{2,1}(u) [z_{1,u}]$$

$$[z_{2,u}] \cdot [w_2] = -\Theta_{2,2}(u) [z_{1,u}]$$

where  $\Theta_{n,k}(t) = \sum_{i \in k + n\mathbb{Z}} t^{i/2n} t^i$ .

Other compositions:  $CF^1(L_{3,u}, L_1) \cong \begin{pmatrix} \bar{z}_u \\ \bar{\xi}_u \end{pmatrix}_{(m_0, 0)}$  choose  $y_{1,u}$  dual to  $z_{1,u}$   
 $CF^1(L_{3,u}, L_2) \cong \begin{pmatrix} \bar{z}_u \\ \bar{\xi}_u \end{pmatrix}_{(m_0, -2m_0)}$  choose  $y_{2,u}$  dual to  $z_{2,u}$

The products  $[y_{1,u}] \cdot [z_{1,u}] \in HF^1(L_1, L_1)$   
 $[z_{2,u}] \cdot [y_{1,u}] \in HF^1(L_{3,u}, L_{3,u})$   
 $[y_{2,u}] \cdot [z_{2,u}] \in HF^1(L_2, L_2)$   
 $[z_{2,u}] \cdot [y_{2,u}] \in HF^1(L_{3,u}, L_{3,u})$

are computed by perturbing one of the legs of the triangle. In these cases, we get the generator of  $HF^1(L, L) \cong H^1(L; \mathbb{R})$  corresponding to the orientation.

From this an associativity of the product on cohomology, we calculate

$$HF^0(L_1, L_2) \otimes HF^1(L_{3,u}, L_1) \rightarrow HF^1(L_{3,u}, L_2)$$

$$[w_1] \cdot [y_{1,u}] = \Theta_{2,1}(u) [y_{2,u}] \quad [w_2] \cdot [y_{1,u}] = -\Theta_{2,2}(u) [y_{2,u}]$$

$$HF^1(L_{3,u}, L_1) \otimes HF^0(L_2, L_{3,u}) \rightarrow HF^1(L_2, L_1)$$

$$[y_{1,u}] \cdot [z_{2,u}] = \Theta_{2,2}(u) [w_3] + \Theta_{2,1}(u) [w_4]$$

Now pick  $u \notin \{ \pm t^{\frac{1}{2}k} \mid k \in \mathbb{Z} \}$  then  $L_{3,u}$  and  $L_{3,u^{-1}}$  are disjoint. So  $H^*(L_{3,u}, L_{3,u^{-1}}) = 0$

Consider the triangle of morphisms.

$$\begin{array}{ccc}
 & c(\Theta_{2,2}(u)[w_1] + \Theta_{2,1}(u)[w_2]) & \\
 L_1 & \xrightarrow{\quad\quad\quad} & L_2 \\
 & \nwarrow [1] & \nearrow \\
 (y_{1,u}, -y_{1,u^{-1}}) & & (z_{2,u}, z_{2,u^{-1}}) \\
 & L_{3,u} \oplus L_{3,u^{-1}} & \\
 & \text{Normalization} & \\
 & c = [\Theta'_{4,3}(1)(\Theta_{4,1}(u) - \Theta_{4,3}(u))]^{-1} & 
 \end{array}$$

Claim: the composition of any two consecutive arrows is zero.

Proof: This follows from the computations of the products above, together with the theta function identity

$$\Theta_{n,k}(t^{-1}) = \Theta_{n,n-k}(t)$$

Claim: the triangle is exact in  $H^0(\text{Fuk}(T)^{\text{TW}})$

Proof: next time. (need a  $\mu^3$ .)

But this shows that if  $\gamma \in CF^0(L_1, L_2)$  is the morphism in the triangle, then  $C_\gamma \cong L_{3,u} \oplus L_{3,u^{-1}}$ .

provided  $u \notin \{ \pm t^{\frac{1}{2}k} \mid k \in \mathbb{Z} \}$ .  
 [So this  $v$  is not a root of  $p \in \text{Sym}^4(V^1)$ ]