Lecture 33 Some twisted complexes over Q_p.

Recall Q:

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[loop left, "\Lambda^0(V)\oplus\Lambda^2(V)"] \arrow[r, bend left, "V \text{ deg } 0"] & \overset{2}{\bullet} \arrow[l, bend left, "V \text{ deg } 1"] \arrow[loop right, "\Lambda^0(V)\oplus\Lambda^2(V)"] \end{tikzcd}$$

For $p \in \mathrm{Sym}^4(V^\vee)$,

Q_p is the unique (up to equivalence) A_∞-structure on Q w/ $\mu^3 \equiv 0$ and with deformation class:

$$HH^4(Q, Q[-2]) \cong \mathrm{Sym}^4(V^\vee)$$
$$[\mu^4] \longleftrightarrow p$$

We can think of Q_p as an A_∞-category with 2 objects X_1, X_2 corresponding to vertices 1 and 2.
so $\hom^0(X_1, X_2) = V \quad \hom^1(X_2, X_1) = V$, etc.

We want to investigate how the choice of p affects the triangulated A_∞-category Q_p^{TW}.

Let $v \in V = \hom^0(X_1, X_2)$ it is closed since $\mu^1 \equiv 0$ on Q_p, so we may form the twisted complex

$$C_v = \mathrm{Cone}(v : X_1 \to X_2) = \left(X_1[1] \oplus X_2,\ \delta = \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix} \right)$$

$$\hom_{Q_p^{TW}}(C_v, C_v) = \begin{pmatrix} \Lambda^0(V)\oplus\Lambda^2(V)[-1] & V \\ V[-1] & \Lambda^0(V)\oplus\Lambda^2(V)[-1] \end{pmatrix}$$

(Direct sum of four vector spaces.)

Since $\mu^1_{Q_p}$ and $\mu^3_{Q_p}$ vanish, the differential $\mu^1_{Q_p^{TW}}$ is

$$\mu^1_{Q_p^{TW}}(x) = \mu^2_{Q_p^{\oplus}}(\delta, x) + \mu^2_{Q_p^{\oplus}}(x, \delta)$$

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \xrightarrow{\mu^1_{Q_p^{TW}}} \mu^2_{Q_p^{\oplus}}\left(\begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix}, \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right) + \mu^2_{Q_p^{\oplus}}\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix}\right)$$

$$= \begin{pmatrix} x_{12} \wedge v & 0 \\ -v \wedge x_{11} - x_{22} \wedge v & v \wedge x_{12} \end{pmatrix}$$

My discussion of signs before was incomplete – sorry.

The cohomology is 4-dimensional, represented by

$$e = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \; t = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}, \; q = \begin{pmatrix} v \wedge v^* & 0 \\ 0 & v \wedge v^* \end{pmatrix}, \; u = \begin{pmatrix} 0 & 0 \\ v^* & 0 \end{pmatrix}$$

where $v^* \in V$ is any element such that $v \wedge v^* \neq 0$.
[e is actually the identity element: sign is due to the shift in $X_1[1] \oplus X_2$.]

The product $\mu^2_{Q_p^{TW}}$ involves $\mu^2_{Q_p^{\oplus}}$ and $\mu^4_{Q_p^{\oplus}}$:

$$\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \otimes \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \xrightarrow{\mu^2_{Q_p^{TW}}} \begin{pmatrix} y_{11} \wedge x_{11} + y_{12} \wedge x_{21} & y_{11} \wedge x_{12} + y_{12} \wedge x_{22} \\ y_{12} \wedge x_{11} + y_{22} \wedge x_{21} & y_{22} \wedge x_{22} + y_{21} \wedge x_{12} \end{pmatrix}$$

$$+ \begin{pmatrix} \mu^4_{Q_p}(y_{12}, v, x_{12}, v) & \mu^4_{Q_p}(v, y_{12}, v, x_{11}) \\ \mu^4_{Q_p}(y_{22}, v, x_{12}, v) + \mu^4_{Q_p}(v, y_{12}, x_{22}, v) + \mu^4_{Q_p}(v, y_{11}, x_{12}, v) & \mu^4(v, y_{12}, v, x_{12}) \end{pmatrix}$$

We find
$$\mu^2_{Q_p^{TW}}(u,t) = \tfrac{1}{2}\mu^1_{Q_p^{TW}}\begin{pmatrix} 0 & v^* \\ 0 & 0 \end{pmatrix} - \tfrac{1}{2} q$$

$$\mu^2_{Q_p^{TW}}(t,u) = \tfrac{1}{2}\mu^1_{Q_p^{TW}}\begin{pmatrix} 0 & v^* \\ 0 & 0 \end{pmatrix} + \tfrac{1}{2} q$$

$$\mu^2_{Q_p^{TW}}(u,u) = 0$$

$$\mu^2_{Q_p^{TW}}(t,t) = p(v) e$$

Taking degree 0 cohomology, which is spanned by e, t, we find

$$H^0(\hom_{Q_p^{TW}}(C_v, C_v)) \cong R[t]/(t^2 - p(v))$$

Now for any fixed $v \in V$, $p(v) \in R$ is just an element of the ground field (which is algebraically closed). there are two cases:

$p(v)=0$: $\quad H^0(\hom_{Q_p^{TW}}(C_v, C_v)) \cong R[t]/(t^2)$ has nilpotent element.

$p(v) \neq 0$: Then $\sqrt{p(v)} \in R$ so we have
$$H^0(\hom_{Q_p^{TW}}(C_v, C_v)) \cong R[t]/(t - \sqrt{p(v)})(t + \sqrt{p(v)}) \cong R \times R$$
is a semisimple ring.

This proves

<u>Lemma</u>: $H^0(\hom_{Q_p^{TW}}(C_v, C_v))$ is semisimple iff $p(v) \neq 0$.

This means we can determine $p \in \mathrm{Sym}^4(V^\vee)$ (up to a scalar)
By figuring out those v for which $H^0(\hom_{Q_p^{TW}}(C_v, C_v))$ is semisimple.

$\circledast$ Now consider the elliptic curve $Y_p = \{y^2 = p(v)\} \subseteq \mathrm{Tot}(\mathcal{O}_{\mathbb{P}(V)}(2) \to \mathbb{P}(V))$
where $p \in \mathrm{Sym}^4(V)$. We took

$$E_1 = \mathcal{O}_{\mathbb{P}(V)} \quad E_2 = \mathcal{O}_{\mathbb{P}(V)}(1) \otimes \Lambda^2(V) \quad \text{and} \quad X_i = \pi^* E_i \in \mathrm{Coh}(Y_p)$$

Then we can form $\quad C_v = \mathrm{Cone}(v : X_1 \to X_2)$.
in $D^b(\mathrm{Coh}\, Y_p)$, there is an exact triangle

$$\begin{tikzcd} \pi^* E_1 \arrow[rr, "v"] & & \pi^* E_2 \arrow[dl] \\ & \mathcal{O}_{\pi^{-1}([v])} \arrow[ul, "{[1]}"] & \end{tikzcd}$$

where $[v] \in \mathbb{P}(V)$ and $\pi^{-1}([v])$ is the scheme-theoretic fiber of π.

This means that in $D^b(\mathrm{Coh}\, Y_p)$, $\quad C_V \cong \mathcal{O}_{\pi^{-1}([V])}$

If $p(V) \neq 0$, then $\pi^{-1}([V])$ consists of two distinct points y_1, y_2:

$$\mathcal{O}_{\pi^{-1}([V])} \cong \mathcal{O}_{y_1} \oplus \mathcal{O}_{y_2}$$

so $\quad \mathrm{End}(\mathcal{O}_{\pi^{-1}([V])}) = \mathrm{End}(\mathcal{O}_{y_1}) \times \mathrm{End}(\mathcal{O}_{y_2}) = R \times R$

is semisimple.

If $p(V) = 0$: then $\pi^{-1}([V])$ is a fat point $\quad \mathrm{End}(\mathcal{O}_{\pi^{-1}([V])}) = R[t]/(t^2)$

If $\tilde{p} \in \mathrm{Sym}^4(V^\vee) \cong HH^4(Q, Q[-2])$ is the A_∞-deformation class of the elliptic curve, these properties imply that p and $\tilde{p}$ differ at most by a scalar factor!

From an A_∞-category $\mathcal{A}$, we formed a triangulated A_∞-category $\mathcal{A}^{tw}$.
There is another enlargement we can do, which is to introduce splittings of all idempotent endomorphisms.

At the cohomological level, this means we consider pairs (X, p), where X is an object and $p^2 = p$ in $\mathrm{Hom}(X, X)$

then we set

$$\mathrm{Hom}((X_0, p_0), (X_1, p_1)) = p_1 \cdot \mathrm{Hom}(X_0, X_1) \cdot p_0$$

The A_∞-version of this construction applied to $\mathcal{A}^{tw}$ yields a triangulated A_∞-category called $\mathcal{A}^{perf}$ or $D^\pi(\mathcal{A})$.

Using the fact that $\pi^* E_1$ and $\pi^* E_2$ "split-generate" $D^b(\mathrm{Coh}\, Y_p)$ means there is a quasi-equivalence $\quad Q_p^{perf} \to D^b(\mathrm{Coh}\, Y_p)$.