Lecture 32 Twisted complexes and triangulated A_∞-categories.

We need a concept from A_∞-category theory, which we could have introduced back when we were doing triangulated categories. It answers the question, how does one form the triangulated envelope of an A_∞-category?

Let $\mathcal{A}$ be an A_∞-category.

Additive enlargement: This is a category $\mathcal{A}^\oplus$ whose objects are formal sums

$$X = \bigoplus_{i\in I} F^i \otimes X^i[-\sigma^i]$$

where I is a finite set, $X^i \in \mathrm{Ob}\,\mathcal{A}$, F^i vector spaces, and $\sigma^i \in \mathbb{Z}$ are formal grading shifts. The morphism spaces are

$$\hom^p_{\mathcal{A}^\oplus}\Big(\bigoplus_{i\in I} F^i \otimes X^i[-\sigma^i], \bigoplus_{j\in J} F^j \otimes X^j[-\sigma^j]\Big)$$

$$= \bigoplus_{i,j} \hom^{p+\sigma^i-\sigma^j}_{\mathcal{A}}(X_i, X_j) \otimes \mathrm{Hom}(F^i, F^j)$$

The operations $\mu^d_{\mathcal{A}^\oplus}$ combine $\mu^d_{\mathcal{A}}$ with "matrix multiplication" and composition of linear maps.

Twisted complexes: A twisted complex is a pair (X, δ_X) where $X \in \mathrm{Ob}\,\mathcal{A}^\oplus$ and $\delta_X \in \hom^1_{\mathcal{A}^\oplus}(X,X)$

satisfying the Maurer-Cartan equation

$$0 = \sum_{d=1}^{\infty} \mu^d_{\mathcal{A}^\oplus}(\delta_X^{\otimes d}) = \mu^1_{\mathcal{A}^\oplus}(\delta_X) + \mu^2_{\mathcal{A}^\oplus}(\delta_X, \delta_X) + \mu^3_{\mathcal{A}^\oplus}(\delta_X, \delta_X, \delta_X) + \cdots$$

To gaurentee converges, we require δ_X is strictly decreasing w.r.t. a filtration.

These are the objects of $\mathcal{A}^{TW}$. The composition in $\mathcal{A}^{TW}$ uses that of $\mathcal{A}^{\oplus}$, but we "insert δ_X in all possible ways".

$$\mu^d_{\mathcal{A}^{TW}}(a_d,\dots,a_1) = \sum_{i_0,\dots,i_d} \mu_{\mathcal{A}^{\oplus}}^{d+i_0+\dots+i_d}\left(\delta_{X_d}^{\otimes i_d}, a_d, \delta_{X_{d-1}}^{\otimes i_{d-1}},\dots, a_1, \delta_{X_0}^{\otimes i_0}\right)$$

Eg. $\mu^1_{\mathcal{A}^{TW}}(a) = \mu^1_{\mathcal{A}^{\oplus}}(a) + \mu^2_{\mathcal{A}^{\oplus}}(\delta_{X_1}, a) + \mu^2_{\mathcal{A}^{\oplus}}(a, \delta_{X_0}) + \mu^3_{\mathcal{A}^{\oplus}}(\delta_{X_1}, a, \delta_{X_0}) + \cdots$

The importance is that $\mathcal{A}^{TW}$ has mapping cones:

Let (Y_0, δ_{Y_0}) and (Y_1, δ_{Y_1}) be two twisted complexes, and let $c \in \hom^0_{\mathcal{A}^{TW}}(Y_0, Y_1)$ be a degree zero cocycle, $\mu^1_{\mathcal{A}^{TW}}(c) = 0$.

Then $C = \mathrm{Cone}(c)$ is defined to be

$$\left(C = Y_0[1] \oplus Y_1 \;,\; \delta_C = \begin{pmatrix} \delta_{Y_0} & 0 \\ -c & \delta_{Y_1} \end{pmatrix} \right)$$

As expected, cones correspond to exact triangles. An interesting feature of the A_∞-context is that we can define exact triangles without reference to cones.

Let $\mathcal{D}$ be the A_∞-category with objects Z_0, Z_1, Z_2:

$$\begin{tikzcd} Z_0 \arrow[loop left, "e_{Z_0}"] \arrow[rr, "x_1"] & & Z_1 \arrow[loop right, "e_{Z_1}"] \arrow[dl, "x_2"] \\ & Z_2 \arrow[ul, "x_3"] \arrow[loop right, "e_{Z_2}"] & \end{tikzcd}$$

$|x_1| = |x_2| = 0$

$|x_3| = 1$

Where the only nontrivial compositions are

$$\mu^3_{\mathcal{D}}(x_3, x_2, x_1) = e_{z_0}$$
$$\mu^3_{\mathcal{D}}(x_1, x_3, x_2) = e_{z_1}$$
$$\mu^3_{\mathcal{D}}(x_2, x_1, x_3) = e_{z_2}$$

Let $\mathcal{A}$ be an A_∞-category, $H(\mathcal{A})$ the cohomology category. An <u>exact triangle</u> is a diagram in $H(\mathcal{A})$

$$\begin{tikzcd} Y_0 \arrow[rr, "{[c_1]}"] & & Y_1 \arrow[dl, "{[c_2]}"] \\ & Y_2 \arrow[ul, "{[c_3]}", "{[1]}"'] & \end{tikzcd}$$

Such that there is a functor $\mathcal{F}: \mathcal{D} \to \mathcal{A}$ such that $\mathcal{F}(z_i) = Y_i$ $[\mathcal{F}^1(x_i)] = [c_i]$.

$\mathcal{A}$ is called <u>triangulated</u> if it has a shift functor and every $[c_1]$ in $H^0(\mathcal{A})$ can be completed to an exact triangle.

<u>Proposition</u> If $\mathcal{A}$ is a triangulated A_∞-category, its homotopy category $H^0(\mathcal{A})$ is triangulated in the original sense. If $\mathcal{A}$ and $\mathcal{B}$ are triangulated A_∞-categories and $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ is an A_∞-functor, $H^0(\mathcal{F}): H^0(\mathcal{A}) \to H^0(\mathcal{B})$ is an exact functor.

<u>Proposition</u> $\mathcal{A}^{TW}$ is a triangulated A_∞-category with an embedding $\mathcal{A} \to \mathcal{A}^{TW}$, and $\mathcal{A}^{TW}$ is initial with respect to these properties.