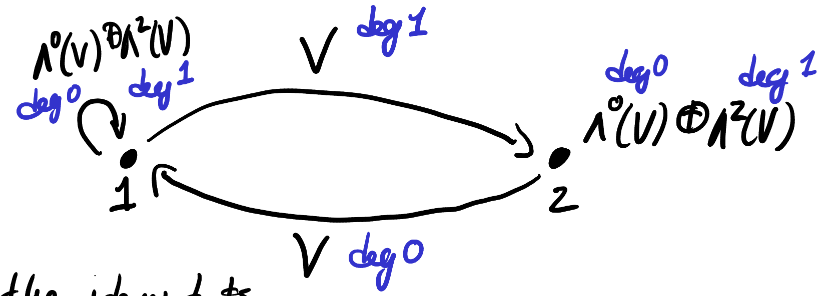
Lecture 31 A_∞-structures on Q

Recall the algebra Q :



Let $e_1 \in \Lambda^0(V)$ and e_2 be the idempotents at the vertices. Q is an algebra over $R_2 = Re_1 \times Re_2$ (R=ground field.)

The Hochschild cohomology of Q can be calculated we need

$$(*) \quad HH^d(Q, Q[2-d]) = \begin{cases} 0 & d=3 \\ \mathrm{Sym}^4(V^\vee) & d=4 \\ 0 & d \geq 5 \end{cases}$$

$$(**) \quad HH^d(Q, Q[3-d]) = 0 \quad \text{for } d \geq 7$$

Remarks on these calculations: These use the theory of Koszul algebras. One can observe that Q is isomorphic as an algebra to $\Lambda^\bullet(V) \rtimes \mathbb{Z}_2$ where the nontrivial $\tau \in \mathbb{Z}_2$ acts as -1 on V.
(This isomorphism is <u>not</u> compatible with the grading, so care is needed.)
This algebra is Koszul, and its Koszul dual is $\mathrm{Sym}^\bullet(V^\vee) \rtimes \mathbb{Z}_2$. (Compare the classical Koszul duality of exterior and symmetric algebras)
The nontrivial $HH^4(Q, Q[-2]) \cong \mathrm{Sym}^4(V^\vee)$ corresponds under Koszul duality to the degree 4 part of the center of $\mathrm{Sym}^\bullet(V^\vee) \rtimes \mathbb{Z}_2$.

So consider an A_∞-structure $(\mu^d)_{d\geq 1}$ on Q starting with that respects the grading and the R_2-algebra structure.

Because $HH^3(Q, Q[-1]) = 0$, after a gauge transformation we may assume $\mu^3 \equiv 0$.

As part of μ^4, we have maps

$$\mu^4_{[12121]} : e_1Qe_2 \otimes e_2Qe_1 \otimes e_1Qe_2 \otimes e_2Qe_1 \cong V^{\otimes 4} \to Re_1 \cong R$$
$$\mu^4_{[21212]} : \quad \text{similar} \quad \text{———————} \cong V^{\otimes 4} \to Re_2 \cong R$$

Using $\mu^3 \equiv 0$ and the A_∞-equations, one can see that these two functions are equal:

$$p(v) = \mu^4_{[12121]}(v,v,v,v) = \mu^4_{[21212]}(v,v,v,v) \quad , \quad p \in Sym^4(V^\vee)$$

In fact, under the isomorphism

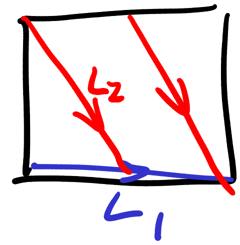
$$HH^4(Q, Q[-2]) \cong Sym^4(V^\vee)$$
$$[\mu^4] \longleftrightarrow p$$

Since $HH^d(Q, Q[2-d]) = 0$ for $d \geq 5$, this class $p = [\mu^4]$ determines the A_∞-structure up to gauge equivalence.

The fact $HH^d(Q, Q[3-d])$, $d \geq 7$, is used to show that an A_∞-structure exists for any given p.

<u>Definition</u> Let Q_p denote Q with an A_∞-structure whose class $[\mu^4] = p$.

Now consider the two-torus



$$Q = \bigoplus_{i,j=1}^{2} HF(L_i, L_j)$$

There is a nontrivial A_∞-structure on Q determined by the pseudo-holomorphic maps in T.
By the classification theory, this is determined by a $p \in \mathrm{Sym}^4(V^\vee)$. What p is it???

Seidel finds the answer, and its not particularly simple:

Recall notation: $R = \left\{ u = \sum_{k=0}^{\infty} c_k \hbar^{m_k} \;\middle|\; c_k \in \mathbb{C}, m_k \in \mathbb{R}, \lim_{k\to\infty} m_k = +\infty \right\}$

Novikov field

$$F = \left\{ f(t) = \sum_{k=0}^{\infty} c_k \hbar^{m_k} t^{n_k} \;\middle|\; c_k \in \mathbb{C}, m_k \in \mathbb{R}, n_k \in \mathbb{Z}, \lim_{k\to\infty} m_k + A n_k = +\infty \ \forall A \in \mathbb{R} \right\}$$

$$\theta_{n,k}(t) \in F, \quad \theta_{n,k}(t) = \sum_{i \in n\mathbb{Z}+k} \hbar^{i^2/2} t^i$$

Define the <u>unit torus polynomial</u> $p \in R[V_1, V_2]$

$$p(V_1,V_2) = c\left(\theta_{2,2}(\hbar^{1/2})^2 V_2^2 - \theta_{2,1}(\hbar^{1/2})^2 V_1^2\right) \cdot \left(\theta_{2,2}(1)^2 V_2^2 - \theta_{2,1}(1)^2 V_1^2\right)$$

where $c = -\hbar^{1/4}\, \theta_{1,1}(1)^{-2}\, \theta_{1,1}(-1)^{-2}\, \theta_{1,1}(\hbar^{1/2})^{-2}\, \theta'_{4,3}(1)^2$

This is the unique (up to scaling) homogeneous quartic polynomial that vanishes at the four points

$$[\Theta_{2,2}(\pm\hbar^{1/2}) : \Theta_{2,1}(\pm\hbar^{1/2})] \,,\; [\Theta_{2,2}(\pm 1) : \Theta_{2,1}(\pm 1)]$$

in $\mathbb{P}^1_{\mathbb{R}} \cong \mathbb{P}(V)$

On the algebraic side, recall the double cover

$$Y_p \xrightarrow{\pi} \mathbb{P}(V) \qquad Y_p = \{ y^2 = p(v) \}$$

$$E_1 = \mathcal{O}_{\mathbb{P}(V)} \qquad E_2 = \mathcal{O}_{\mathbb{P}(V)}(1) \otimes \Lambda^2(V)$$

$$Q \cong \bigoplus_{i,j=1}^{2} \mathrm{Ext}^{\bullet}_{Y_p}(\pi^* E_i, \pi^* E_j)$$

This algebra also inherits an A_∞-structure from a DG-enhancement of the derived category $D^b(\mathrm{Coh}\, Y_p)$.

It is indeed equivalent to Q_p, where $p \in \mathrm{Sym}^4(V^\vee)$ is the polynomial defining Y_p.