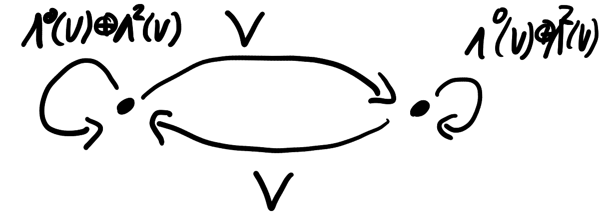
# Lecture 30 Deformation and classification of $A_\infty$-structures




We have calculated the graded algebra $Q$ associated to our pair of Lagrangians in $T$

We still need to figure out the $A_\infty$-structure. This is difficult to calculate directly since we have to perturb the Lagrangians several times. Another approach is to try to find invariants of $A_\infty$ structures on $Q$ that classify them.

<u>General theory</u>: Let $A$ be a graded algebra over a field $\mathbb{K}$ (char 0)

Let $\mathcal{U}(A)$ be the set of $A_\infty$-structures on $A$, $\{\mu^d_{\mathcal{A}}\}_{d\geq 1}$ $\mu^d : A^{\otimes d} \to A[2-d]$ such that

$$\mu^1_{\mathcal{A}} = 0 \qquad \mu^2_{\mathcal{A}}(a_2, a_1) = (-1)^{|a_1|} a_2 a_1$$

($A_\infty$-equations) $$\sum_{e,i} (-1)^* \mu_{\mathcal{A}}^{d-e+1}(a_d, \ldots, a_{i+e+1}, \mu^e_{\mathcal{A}}(a_{i+e}, \ldots, a_{i+1}), a_i, \ldots, a_1) = 0$$
$$* = |a_1| + \cdots + |a_i| - i$$

This means that $H(\mathcal{A}) = A$ as graded algebras.

Recall that an $A_\infty$-homomorphism $\mathcal{G}: \mathcal{A} \to \mathcal{B}$ is a sequence of maps
$$(\mathcal{G}^1, \mathcal{G}^2, \ldots) \qquad \mathcal{G}^d : \mathcal{A}^{\otimes d} \to \mathcal{B}[1-d]$$
such that
$$\sum_r \sum_{s_1 \cdots s_r} \mu^r_{\mathcal{B}}(\mathcal{G}^{s_r}(a_d, \ldots, a_{d-s_r+1}), \ldots, \mathcal{G}^{s_1}(a_{s_1}, \ldots, a_1))$$
$$= \sum_{m,n} (-1)^* \mathcal{G}^{d-m+1}(a_d, \ldots, a_{n+m+1}, \mu^m_{\mathcal{A}}(a_{n+m}, \ldots, a_{n+1}), a_n, \ldots, a_1)$$

Two elements $\mathcal{A}, \mathcal{A}' \in \mathcal{U}(A)$ are <u>equivalent</u> is there is an $A_\infty$-morphism $G : \mathcal{A} \to \mathcal{A}'$ such that $G^1 = Id_A$.

In this case, $\mathcal{A}'$ is determined by $\mathcal{A}$ and $G$, for instance

$$\begin{aligned}
&\mu^3_{\mathcal{A}'}(a_3, a_2, a_1) \\
&= \mu^3_{\mathcal{A}}(a_3, a_2, a_1) + G^2(a_3, \mu^2_{\mathcal{A}}(a_2, a_1)) \\
&+ (-1)^{|a_1|-1} G^2(\mu^2_{\mathcal{A}}(a_3, a_2), a_1) - \mu^2_{\mathcal{A}}(a_3, G^2(a_2, a_1)) \\
&- \mu^2_{\mathcal{A}}(G^2(a_3, a_2), a_1)
\end{aligned}$$

And there are no constraints on what $G$ can be.
This means that the set

$$\mathcal{G}(A) = \{ (G^1 = Id_A, G^2, G^3, \dots) \mid G^d : A^{\otimes d} \to A[1-d] \}$$

acts on $\mathcal{U}(A)$.

$\mathcal{G}(A)$ is called the <u>Gauge group</u>, the group law is

$$(G \circ F)^d(a_d, \dots, a_1) = \sum_r \sum_{s_1 \cdots s_r} G^r(F^{s_r}(a\text{'s}), \dots, F^{s_1}(a_{s_1}, \dots, a_1))$$

$\mathcal{U}(A)/\mathcal{G}(A)$ is the moduli space of $A_\infty$-structures extending the given product.

There is also an action of the monoid $(\mathbb{K}, \cdot)$ on $\mathcal{U}(A)/\mathcal{G}(A)$: multiply $\mu^d$ by $\varepsilon^{d-2}$ for $\varepsilon \in \mathbb{K}$.

The fixed point ($\mu^d = 0$ for $d \geq 3$) is the <u>formal</u> $A_\infty$-structure.

We wish to understand the "tangent space" at this point.

This is described in terms of the Hochschild cohomology of $A$.

Hochschild cohomology: Given an associative algebra $A$, we can consider the category of $A$-bimodules. An $A$-bimodule $M$ has a left $A$-action and a right $A$-action which commute. This is the same as a left $A \otimes A^{op}$-action.

We define
$$HH^i(A,M) = \mathrm{Ext}^i_{A\otimes A^{op}}(A,M)$$

In particular, taking $M=A$ we get $HH^i(A,A)$.
There is a complex computing this with cochains
$$CC^i(A,M) = \mathrm{Hom}_{\mathbb{K}}(A^{\otimes i}, M)$$

$$(d\psi)(x_0,\dots,x_n) = x_0\psi(x_1,\dots,x_n) + \sum_{i=1}^{n-1}(-1)^i \psi(x_0,\dots,x_{i-1}x_i,\dots,x_n) + (-1)^n \psi(x_0,\dots,x_{n-1})x_n$$

for $\psi \in \mathrm{Hom}_{\mathbb{K}}(A^{\otimes n}, M)$.

When $A$ is a graded algebra we take everything in the graded sense. In this case $A[j]$ is a different bimodule, we twist the right action:
$$m \cdot a = (-1)^{j|m||a|} ma$$
and we get a bigraded Hochschild cohomology

$$HH^i(A,A[j]) = \mathrm{Ext}^i_{A\otimes A^{op}}(A,A[j])$$

A cochain here is $\psi \in CC^i(A,A[j]) = \mathrm{Hom}_{\mathbb{K}}(A^{\otimes d}, A[j])$

Observe that the $\mu^d$ maps of an $A_\infty$-structure lie in
$$\mu^d_A \in \mathrm{Hom}_{\mathbb{K}}(A^d, A[2-d]) = CC^d(A, A[2-d]).$$

**Proposition** Let $\mathcal{A} \in \mathcal{U}(A)$ be such that $\mu^i_{\mathcal{A}} \equiv 0$ for $i \in \{3, \ldots, d-1\}$

Then $\mu^d_{\mathcal{A}}$ is a Hochschild cocycle, and represents a class
$$[\mu^d_{\mathcal{A}}] \in HH^d(A, A[2-d])$$

**Proof** The $(d+1)$th $A_\infty$-equation involves $\mu^i_{\mathcal{A}}$, $i \in \{1, 2, \ldots, d\}$. Since all by $\mu^2_{\mathcal{A}}, \mu^d_{\mathcal{A}}$ vanish, the equation reduces to the condition that $\mu^d_{\mathcal{A}}$ is closed.

**Def** In the situation above, we call $[\mu^d_{\mathcal{A}}]$ the order $d$ obstruction class. Then we have

**Prop**: If $[\mu^d_{\mathcal{A}}] = 0$, there is a Gauge transformation $(G^i)$ such that $G \cdot \mathcal{A} = \mathcal{A}'$ has $\mu^i_{\mathcal{A}'} \equiv 0$ for $i \in \{3, \ldots, d\}$.

Consider the case where $HH^d(A, A[2-d]) = 0$ for all $d \geq 3$. Then $\mathcal{U}(A)/\mathcal{G}(A)$ reduces to a point, represented by the formal $A_\infty$-structure. In this case $A$ is said to be <u>intrinsically formal</u>.

In the case of the quiver algebra $Q$ above, we shall find that

$$HH^d(Q, Q[2-d]) = \begin{cases} 0 & d = 3 \\ \mathrm{Sym}^4(V^\vee) & d = 4 \\ 0 & d \geq 5 \end{cases}$$

So $A_\infty$-structures are classified by an element of $\mathrm{Sym}^4(V^\vee)$, a quartic polynomial in 2 variables!