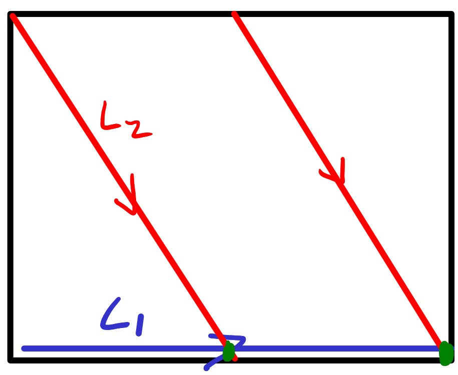
Lecture 29 A quiver algebra from the two-torus.

Let R be our ground field.

Consider the two torus $T = \mathbb{R}^2/\mathbb{Z}^2$ $\omega = dp \wedge dq$ $\eta = dz = dp + i\,dq$ as before. We investigate the total endomorphism algebra of a particularly chosen pair of objects.



$L_1 = \{q = 0\}$

$L_2 = \{q = -2p\}$

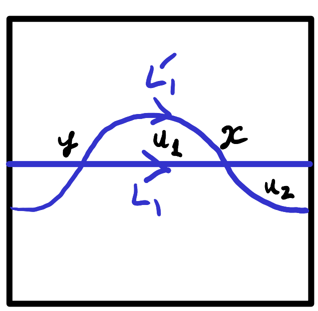
Same conventions for brane structure as in last lecture.

$w_1 \quad , \quad -w_2 \in CF^0(L_1, L_2)$

$w_4 \quad , \quad w_3 \in CF^1(L_2, L_1)$

We declare that $(\frac{1}{2}, 0)$ represents $w_1 \in CF^0(L_1, L_2)$ and $w_4 \in CF^1(L_2, L_1)$
$(0, 0)$ represents $-w_2 \in CF^0(L_1, L_2)$ and $w_3 \in CF^1(L_2, L_1)$

The self-Floer cohomology of L_1 is $HF^*(L_1, L_1) \cong H^*(L_1; R)$
To see this, we perturb L_1 off itself



$CF^*(L_1, L_1) = CF^*(L_1, L_1')$

$\deg(x) = 0 \qquad \deg(y) = 1$

$\partial y = 0 \qquad \partial x = (|c_{u_1}| + |c_{u_2}|)\,y = (-1+1)\,y = 0$

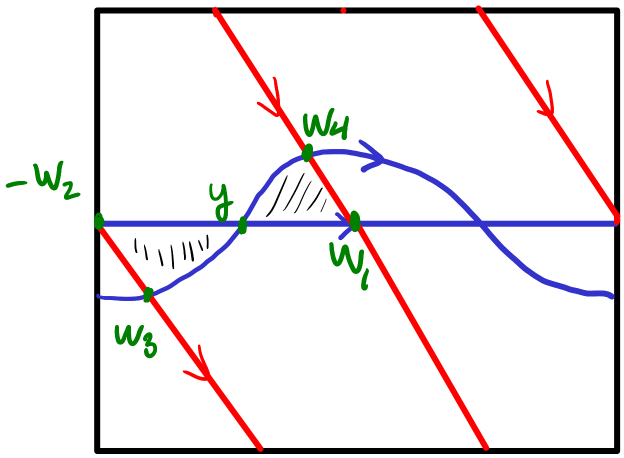
So $HF^0(L_1, L_1) = Rx$
$HF^1(L_1, L_1) = Ry$

Similarly, $HF^*(L_2, L_2) \cong H^*(L_2; R)$

Thus $\bigoplus_{i,j=1}^{2} HF^*(L_i, L_j)$ is an 8-dimensional algebra.

To compute the product: $HF^0(L_1, L_2) \otimes HF^1(L_2, L_1) \to HF^1(L_1, L_1)$

we use L_1': $HF^0(L_1, L_2) \otimes HF^1(L_2, L_1') \to HF^1(L_1, L_1')$



These triangles show

$$[w_4]\cdot[w_1] = [y] \in H^1(L_1; R)$$
$$-[w_3]\cdot[w_2] = [y] \in H^1(L_1; R)$$

Using perturbed copy of L_2, can also see

$$\left.\begin{array}{l}[w_1]\cdot[w_4] \\ -[w_2]\cdot[w_3]\end{array}\right\} \in H^1(L_2; R)$$

equal the generator determined by the orientation of L_2.

For $HF^0(L_1, L_1) \otimes HF^1(L_2, L_1) \to HF^1(L_2, L_1)$

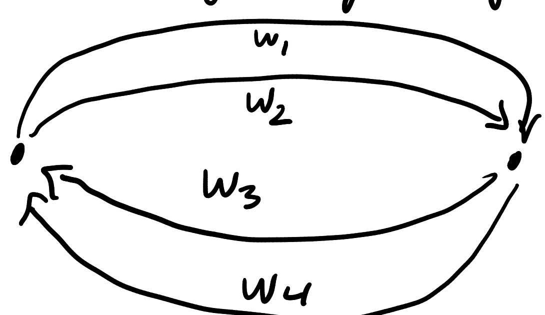
$HF^0(L_1, L_2) \otimes HF^0(L_1, L_1) \to HF^0(L_1, L_2)$

$[x] \in HF^0(L_1, L_1)$ acts as identity.

similarly, $HF^0(L_2, L_2)$ acts as identity

Lastly, product with $HF^1(L_i, L_i)$ is zero for degree reasons.

<u>Def</u>: Let Q be the algebra defined by taking the path algebra of the quiver



modulo the relations

$$w_3 w_2 + w_4 w_1 = 0,\ w_1 w_4 + w_2 w_3 = 0,\ w_3 w_1 = 0,\ w_4 w_2 = 0.$$

<u>Proposition</u> $\displaystyle\bigoplus_{i,j=1}^{2} HF(L_i, L_j) \cong Q.$

Why did we choose these objects?

1. We shall see that L_1 and L_2 generate the Fukaya category.
2. We can find the same algebra coming from coherent sheaves on an elliptic curve.

A slightly more abstract description of Q:

Let V be a 2-dimensional $\mathbb{R}$-vector space. Then define a category with two objects $1, 2$ $\quad \mathrm{Hom}^0(1,2) = V \quad \mathrm{Hom}^1(2,1) = V$

$$\mathrm{Hom}^*(1,1) = \mathrm{Hom}^*(2,2) = \underset{\deg 0}{\Lambda^0(V)} \oplus \underset{\deg 1}{\Lambda^2(V)}$$

the composition is wedge product.

Then $Q \cong \bigoplus_{i,j=1}^{2} \mathrm{Hom}^*(i,j)$

<u>Elliptic curves</u>: Let V be a two-dimensional vector space.

$\mathbb{P}(V)$ = space of lines in V.

An element $p \in \mathrm{Sym}^4(V^\vee)$ defines a quartic polynomial on V.
Assume p vanishes at 4 distinct points of $\mathbb{P}(V)$.

Let Y_p be the double covering of $\mathbb{P}(V)$ consisting of the square roots of p. In local coordinate z on $\mathbb{P}(V)$

$$p(z) = a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4$$

$$Y_p \supseteq \{ y^2 = p(z) \} \leftarrow \text{this is an affine curve.}$$

More invariantly, Y_p is a subvariety of the total space of the line bundle $\mathcal{O}_{\mathbb{P}(V)}(2)$

$$Y_p \subseteq \mathrm{Tot}(\mathcal{O}_{\mathbb{P}(V)}(2) \to \mathbb{P}(V))$$

It is a smooth genus 1 curve, and there is a 2:1 morphism $\pi : Y_p \to \mathbb{P}(V)$ branched over the zeros of p.

Let $E_1 = \mathcal{O}_{\mathbb{P}(V)}$ and $E_2 = \mathcal{O}_{\mathbb{P}(V)}(1) \otimes \Lambda^2(V)$, line bundles on $\mathbb{P}(V)$.

Then $\mathrm{Hom}_{\mathbb{P}(V)}(E_1, E_2) = V^\vee \otimes \Lambda^2(V) \cong V$

$\mathrm{Ext}^1_{\mathbb{P}(V)}(E_1, E_2) = 0$.

Pulling back to Y_p and using $\pi_* \mathcal{O}_{Y_p} \cong \mathcal{O}_{\mathbb{P}(V)} \oplus \mathcal{O}_{\mathbb{P}(V)}(-2)$

$$\cong \mathcal{O}_{\mathbb{P}(V)} \oplus (\Omega^1_{\mathbb{P}(V)} \otimes \Lambda^2(V))$$

$$\begin{aligned}
\mathrm{Ext}^*_{Y_p}(\pi^*E_1, \pi^*E_2) &\cong H^*(Y_p, \pi^*E_1^\vee \otimes \pi^*E_2) \\
&\cong H^*(Y_p, \pi^*(E_1^\vee \otimes E_2)) \\
&\cong H^*(\mathbb{P}(V), (E_1^\vee \otimes E_2) \otimes \pi_*\mathcal{O}_{Y_p}) \quad \text{projection formula} \\
&\cong H^*(\mathbb{P}(V), E_1^\vee \otimes E_2) \oplus H^*(\mathbb{P}(V), E_1^\vee \otimes E_2 \otimes \Omega^1_{\mathbb{P}(V)}) \otimes \Lambda^2(V) \\
&\cong \mathrm{Ext}^*_{\mathbb{P}(V)}(E_1, E_2) \oplus \mathrm{Ext}^{1-*}_{\mathbb{P}(V)}(E_2, E_1)^\vee \otimes \Lambda^2(V). \\
&\cong V \text{ in degree } 0.
\end{aligned}$$

Swapping roles of E_1 and E_2, $\mathrm{Ext}^1_{Y_p}(\pi^*E_2, \pi^*E_1) = V^\vee \otimes \Lambda^2(V) \cong V$

And $\mathrm{Ext}^*_{Y_p}(\pi^*E_i, \pi^*E_i) = \underset{\deg 0}{\Lambda^0(V)} \oplus \underset{\deg 1}{\Lambda^2(V)}$

<u>Proposition</u>

$$\mathrm{Ext}^*_{Y_p}(\pi^*E_1 \oplus \pi^*E_2, \pi^*E_1 \oplus \pi^*E_2) \cong Q.$$

So the same algebra appears here!
Now there is actually a non-trivial A_∞-structure in both cases (T and Y_p). In the case of Y_p, the equivalence class of the A_∞-structure <u>depends on</u> p. The next task is to understand this dependence.