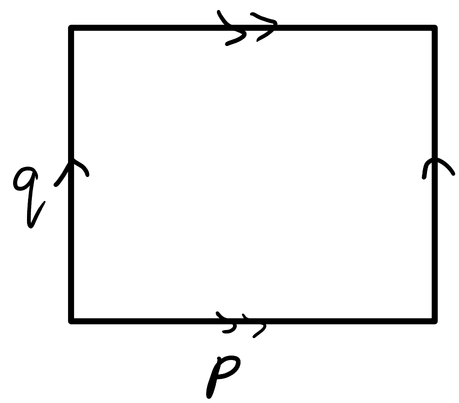
Lecture 28 The two-torus

We consider homological mirror symmetry for the two-torus. This case was originally considered by Kontsevich in the paper where he proposed HMS for the first time. Polishchuk - Zaslow provided the first proof. The proof I will present is due to Seidel ("Abstract analogues of flux as symplectic invariants")

<u>An example computation</u>: To get a feeling for the ingredients, let us begin with a single computation that shows some key features.

Let $T = \mathbb{R}^2/\mathbb{Z}^2$ coordinates (p,q)
$\omega_T = dp \wedge dq$ symplectic form

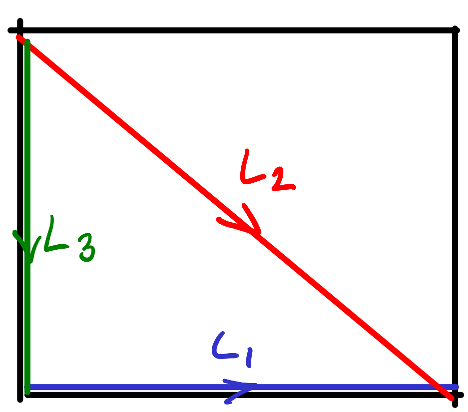


$\eta = dz = dp + i dq$ is a trivializing section of Ω^1_T

(This defines the grading $\eta^2 = dz^2$)

We consider the following Lagrangians.

$L_1 = \{q=0\}$ $L_2 = \{q=-p\}$ $L_3 = \{p=0\}$



We take orientations as in picture, and we give each the trivial Spin structure (orientation trivializes the tangent bundle)

It is possible to choose the gradings so that each of intersection points $x \in CF(L_1, L_2)$ $y \in CF(L_2, L_3)$ $z \in CF(L_1, L_3)$ represent degree zero morphisms, and we do so.

[Note that x, y, z are the same point of T, viz. $(0,0)$]

Technically, the intersection points contribute orientation lines $o(x)$, $o(y)$, $o(z)$ to the Floer complexes. However, in dimension 2 only, it is possible to construct a canonical isomorphism

$$o(x) = \begin{cases} \mathbb{R}, & \deg(x) \text{ even} \\ (TL_1)_x, & \deg(x) \text{ odd} \end{cases} \qquad \underline{\text{for } x : L_0 \to L_1}$$

so we can suppress the orientation lines. (FCPLT §13)

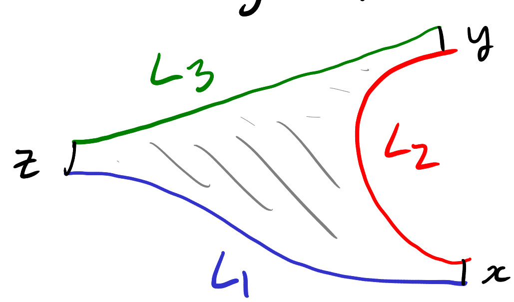
Note there is no differential on $CF(L_1,L_2)$ $CF(L_2,L_3)$ $CF(L_1,L_3)$ so we can identify these with their cohomology.

Now we want to compute the triangle product

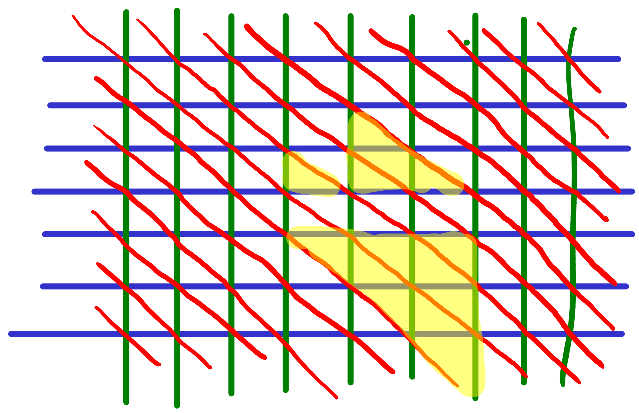
$$\mu^2 : CF^0(L_2,L_3) \otimes CF^0(L_1,L_2) \to CF^0(L_1,L_3)$$
$$(y, x) \longmapsto \mu^2(y,x) = C(x,y,z)\, z$$

Because the spaces are 1-dim'l just need the constant $C(x,y,z)$

It counts triangles



Now we see a problem: there are infinitely many such triangles!
On universal cover of T:



Triangles that differ by translation are identified, but there are still ∞ many different "sizes".

The way to handle this is to introduce a particular coefficient field

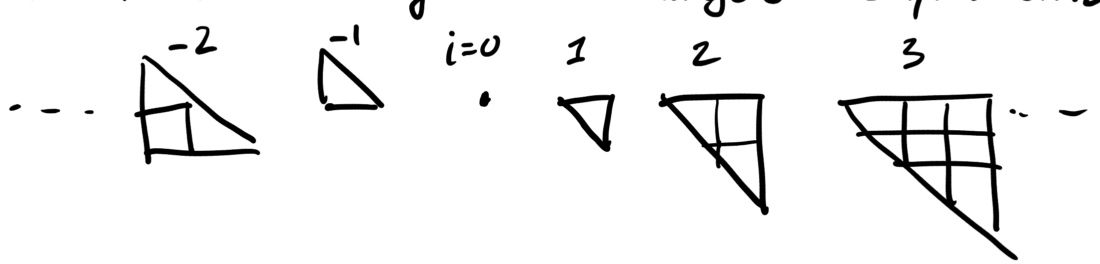
$$R = \left\{ c_0 \hbar^{m_0} + c_1 \hbar^{m_1} + \cdots \;\middle|\; \begin{array}{l} c_k \in \mathbb{C}, \; m_k \in \mathbb{R} \\ \lim_{k\to\infty} m_k = +\infty \end{array} \right\}$$

this is known as the complex Novikov field. $\hbar$ is a formal variable. R is algebraically closed of characteristic 0. When defining any operation, we weight the contribution of the map $u: S \to M$ by an extra factor of

$$\hbar^{\int_S u^* \omega}$$

Then our operations are $\hbar$-adically convergent: Gromov compactness guarantees finiteness below any given area threshhold.

With this definition, $C(x,y,z) \in R$ is a series with one term for each triangle. The triangles are parametrized by $i \in \mathbb{Z}$



They all count positively (I claim) and have area $i^2/2$

so

$$C(x,y,z) = \sum_{i \in \mathbb{Z}} \hbar^{i^2/2}$$

This is a known series, called the theta function.

We shall set the notation

$$\Theta_{n,k}(t) = \sum_{i \in n\mathbb{Z}+k} \hbar^{\frac{i^2}{2n}} t^i$$

This is an element of the $\mathbb{R}$-algebra

$$F = \left\{ f(t) = \sum_{k=0}^{\infty} c_k \hbar^{m_k} t^{n_k} \;\middle|\; \begin{array}{l} c_k \in \mathbb{C},\ m_k \in \mathbb{R},\ n_k \in \mathbb{Z} \\ \lim_{k \to \infty} m_k + A n_k = +\infty \text{ for any } A \in \mathbb{R} \end{array} \right\}$$

The last condition says that $f(a)$ is an $\hbar$-adically convergent series for any $a \in \mathbb{R} \setminus \{0\}$

F is the ring of $\hbar$-adic "analytic functions" on $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$

Thus $C(x,y,z) = \Theta_{1,1}(1)$