Lecture 26 Spin, Pin and orientations of moduli spaces

Spin: For $n \geq 3$, the special orthogonal group $SO(n)$ has $\pi_1(SO(n)) = \mathbb{Z}/2\mathbb{Z}$

The universal covering group $Spin(n) \xrightarrow{2:1} SO(n)$ is called the spin group.

[For $n=2$, $\pi_1(SO(2)) = \mathbb{Z}$, and $Spin(2)$ is the connected 2-fold covering]

[For $n=1$, $SO(1) = \{Id\}$ and $Spin(1) \cong \mathbb{Z}/2\mathbb{Z}$.]

Given an oriented manifold M^n, there is an $SO(n)$-principal bundle associated to the tangent bundle: pick a Riemannian metric, and let $P_{TM} \to M$ be the bundle of oriented orthonormal frames in the tangent spaces.

A spin structure on M is a $Spin(n)$-principal bundle $\tilde{P} \to M$ and an isomorphism of the associated $SO(n)$ bundle with P_{TM}

$$\begin{tikzcd}
\tilde{P} \times_{Spin(n)} SO(n) \arrow[rr, "\sim"] \arrow[dr] & & P_{TM} \arrow[dl] \\
& M &
\end{tikzcd}$$

Here $Spin(n)$ acts on $SO(n)$ via the homomorphism $Spin(n) \to SO(n)$ and left multiplication.

The obstruction to the existence of an orientation is the first Stiefel-Whitney class $w_1(TM) \in H^1(M; \mathbb{Z}_2)$

When $w_1(TM) = 0$, and we pick an orientation, the obstruction to the existence of a Spin structure is $w_2(TM) \in H^2(M, \mathbb{Z}_2)$

There is also a version of Spin that does not depend on the orientability

$$\begin{tikzcd}
Spin(n) \arrow[r] \arrow[d] & Pin(n) \arrow[d] \\
SO(n) \arrow[r] & O(n)
\end{tikzcd}$$

Here $\mathrm{Pin}(n)$ is a 2:1 covering of $O(n)$.
$O(n)$ has 2 connected components and so does $\mathrm{Pin}(n)$
The identity component of $\mathrm{Pin}(n)$ is $\mathrm{Spin}(n)$. There is an exact sequence $1 \to \mathrm{Spin}(n) \to \mathrm{Pin}(n) \to \mathbb{Z}_2 \to 1$
There are actually 2 extensions of $\mathrm{Spin}(n)$ by $\mathbb{Z}_2$.

The $\mathrm{Pin}(n)$ we want has the property that preimages of reflections in $O(n)$ have order 2 in $\mathrm{Pin}(n)$.
[In the other version, they have order 4].

On a manifold M, a Pin structure is a choice of $\mathrm{Pin}(n)$-principal bundle $P^\# \to M$ and an isomorphism

$$\begin{tikzcd} P^\# \times_{\mathrm{Pin}(n)} O(n) \arrow[dr] & \simeq & P_{TM, O(n)} \arrow[dl] \\ & M & \end{tikzcd}$$

unoriented orthonormal frame bundle of TM.

The obstruction to the existence of a Pin structure in this sense is $w_2(TM) \in H^2(M, \mathbb{Z}_2)$.

The purpose of Pin structures is that they are what we need on our Lagrangians in order to fix orientations of the moduli spaces of holomorphic curves with boundary.

A <u>Lagrangian brane</u> in (M, ω, η^2) is a triple $(L, \alpha^\#, P^\#)$ where
- L is a Lagrangian submanifold
- $\alpha^\# : L \to \mathbb{R}$ is a grading: $\exp(2\pi i \alpha^\#) = \alpha(TL)$
- $P^\#$ is a Pin structure on TL.

$\alpha : Gr(TM) \to S^1$ determined by η^2

Determinant lines: Convention: if A and B are one-dimensional real vector spaces, we will write $A \cong B$ ("A and B are canonically isomorphic") to mean that there is an isomorphism $A \to B$ that is determined up to scaling by a positive number.

We denote the dual of V by $V^\vee$.
We denote $\Lambda^{top} V = \Lambda_{\mathbb{R}}^{\dim(V)} V$

Determinant line: Let $D: V \to W$ be a Fredholm operator between real Banach spaces. Set

$$\det(D) = \Lambda^{top}(\mathrm{coker}(D)^\vee) \otimes \Lambda^{top}(\ker(D)).$$

Suppose $\mathcal{M}_S(\{y_j\})$ is one of the moduli spaces we have been considering. If it is regular (Linearized operator $D_{S,u}$ is surjective at each $u \in \mathcal{M}_S(\{y_j\})$) then

$$T\mathcal{M}_S(\{y_j\}) \cong \ker(D_{S,u})$$

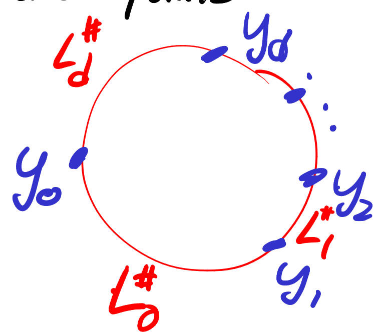
$$\Lambda^{top} T\mathcal{M}_S(\{y_j\}) = \Lambda^{top}\ker(D_{S,u}) \cong \det(D_{S,u})$$

That is, the fibers of the orientation line bundle are determinant lines of the linearized operators.

Given two Lagrangian branes $L_i^\# = (L_i, \alpha_i^\#, P_i^\#)$ $i = 0, 1$, there is an abstract 1-dimensional real vector space $o(y)$ associated to any $y \in \mathcal{C}(L_0, L_1)$ (any Floer datum)

I will not explain this here – see Seidel's book section 11.
$o(y)$ is defined as the determinant of an operator associated to y.

The outcome is that, if S is a disk with $d+1$ marked points (as in μ^d), and we choose brane structures for all Lags,



$$(\Lambda^{top}\mathcal{M}_S(y_0,\dots,y_d))_u$$

$$\cong \det(D_{S,u})$$

$$\cong \underbrace{o(y_0)\otimes o(y_1)^\vee \otimes \cdots \otimes o(y_d)^\vee}$$

this is independent of u, so $\mathcal{M}_S(y_0,\dots,y_d)$ is orientable! Further a choice of trivializations $o(y_i)\cong \mathbb{R}$ determines an orientation of $\mathcal{M}_S(y_0,\dots,y_d)$

Now at an isolated point of $\mathcal{M}_S(\{y_j\})$ there is an iso $T\mathcal{M}_S(\{y_j\})\cong 0$ (the 0-dim vector space)

And hence $(\Lambda^{top}T\mathcal{M}_S(\{y_j\}))_u \cong \Lambda^{top} 0 \cong \mathbb{R}$ canonically.

So $o(y_0)\otimes o(y_1)^\vee \otimes \cdots \otimes o(y_d)^\vee \cong \mathbb{R}$ canonically.

So $o(y_0) \cong o(y_d)\otimes \cdots \otimes o(y_1)$ canonically.

In the parametrized case, where we work with the family of domains $\mathcal{S}^{d+1}\to \mathbb{R}^{d+1}$, we have instead

$$(\Lambda^{top}T\mathcal{M}_{\mathcal{S}^{d+1}}(y_0,\dots,y_d))_{(r,u)} \cong (\Lambda^{top}\mathbb{R}^{d+1})_r \otimes o(y_0)\otimes o(y_1)^\vee \otimes\cdots\otimes o(y_d)^\vee$$

So $\mathcal{M}_{\mathcal{S}^{d+1}}$ is canonically oriented relative to $\mathbb{R}^{d+1}$ and $\{o(y_j)\}$.