Lecture 25 Index theory and dimensions of moduli spaces

The reason we got involved in Maslov classes and other Maslov-type indices is because of their connection with the problem of the dimensions of moduli spaces, and this goes through the analytic index theory of Fredholm operators.

Index: Let V, W be Banach spaces. A bounded linear operator $T: V \to W$ is called Fredholm if (1) $T(V) \subseteq W$ is closed

(2) $\ker(T)$ and $\operatorname{coker}(T) = W/T(V)$ are finite dimensional

[Rmk: the second condition implies the first.]

The index of a Fredholm operator T is
$$\operatorname{ind}(T) = \dim(\ker(T)) - \dim(\operatorname{coker}(T)).$$

Key fact: The index is homotopy invariant: If $(T_s)_{s\in[0,1]}$ is a family of Fredholm operators that is continuous, then $\operatorname{ind}(T_0) = \operatorname{ind}(T_s) = \operatorname{ind}(T_1)$

Often, we get Fredholm operators from geometric-analytic problems: M a manifold E, F vector bundles, $T: \Gamma(E) \to \Gamma(F)$ an (elliptic) differential operator. In this case, the index of T on appropriate completions of $\Gamma(E)$ and $\Gamma(F)$ can be computed from certain characteristic classes of (M, E, F). This is known as index theory, and the main theorem in this area is the Atiyah-Singer index theorem.

Examples: . C compact Riemann surface $\quad E = \mathbb{C} \quad F = \Omega^{0,1}$

$$\bar{\partial} : \Gamma(\mathbb{C}) \to \Gamma(\Omega^{0,1})$$

$\mathrm{ind}(\bar{\partial}) = 2 - 2g$, where g is the genus of C.

(note this is the real index $\dim_{\mathbb{R}} \ker \bar{\partial} - \dim_{\mathbb{R}} \mathrm{coker}\, \bar{\partial}$)

. C compact Riemann surface $\quad E = L$ holomorphic line bundle.

$F = \Omega^{0,1} \otimes L$

$$\bar{\partial}_L : \Gamma(L) \to \Gamma(\Omega^{0,1} \otimes L)$$

$$\mathrm{ind}(\bar{\partial}_L) = \langle 2c_1(L), [C] \rangle + 2 - 2g$$

where $c_1(L)$ is the first chern class of L.

This is a form of the Riemann-Roch theorem,

if we realize $\ker(\bar{\partial}_L) \cong H^0(C, L)$

$\mathrm{coker}(\bar{\partial}_L) \cong H^1(C, L)$

so $\dim_{\mathbb{R}} H^0(C,L) - \dim_{\mathbb{R}} H^1(C,L) = \langle 2c_1(L), [C] \rangle + 2 - 2g$

What about if C has boundary? For instance, let $C = D$ be a disk.

we can take $E = \mathbb{C}^n$ to be a trivial hermitian vector bundle.

Let $F \subset E|_{\partial D}$ be a Lagrangian subbundle along the boundary.

Let $\Gamma(E,F)$ be the collection of sections $s: D \to E$

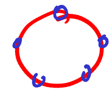
such that $s(p) \in F$ for $p \in \partial D$.

Consider $\bar{\partial}_{E,F} : \Gamma(E,F) \to \Gamma(\Omega^{0,1} \otimes E)$.

Now the family
$$\begin{array}{ccc} F & \subset & E \\ \downarrow & & \downarrow \\ \partial D & \subseteq & D \end{array}$$
defines a loop $\gamma_{E,F} : S^1 \to Gr(\mathbb{C}^n)$ in the Lagrangian grassmannian.

Take $\quad S^1 \xrightarrow{\gamma_{E,F}} Gr(\mathbb{C}^n) \xrightarrow{\det^2} S^1 \quad$ Let $\mu(E,F) = \deg(\det^2 \circ \gamma_{E,F})$

Then $\mathrm{ind}(\bar{\partial}_{E,F}) = \mu(E,F) + n$

$S =$ 

Now for pointed boundary disk $S = \hat{S} \setminus \{\zeta_0, \zeta_1, \dots, \zeta_n\}$
let $\Sigma^- = \{\zeta_0\}$, $\Sigma^+ = \{\zeta_1, \dots, \zeta_n\}$

Let $E = \mathbb{C}^n$ be the trivial bundle over S.
Pick graded Lagrangian boundary conditions $F_i^\#$ over $\partial_i S$
which are constant over the strip like ends.
then at each puncture ζ, we have a pair of graded Lagrangians
$(\Lambda^\#_{\zeta,0}, \Lambda^\#_{\zeta,1})$, and an index $i(\Lambda^\#_{\zeta,0}, \Lambda^\#_{\zeta,1})$.

The Riemann-Roch theorem now says

$$\operatorname{ind}(\bar{\partial}_{E,F}) = i(\Lambda^\#_{\zeta_0,0}, \Lambda^\#_{\zeta_0,1}) - \sum_{j=1}^{n} i(\Lambda^\#_{\zeta_j,0}, \Lambda^\#_{\zeta_j,1}).$$

This now connects to the dimension of moduli spaces.
If $u : S \to M$ is a pseudoholomorphic map with
Lagrangian boundary conditions, we can form

$$\bar{\partial}_u : \Gamma(S, u^*TM, \{u^*TL_i\}) \to \Gamma(\Omega^{0,1}_S \otimes E)$$

The linearization of the pseudo-holomorphic map equation at u.
Then $\operatorname{ind}(\bar{\partial}_u)$ is the "virtual" dimension of the component of
the moduli space containing u. In the transversal case,
$\operatorname{ind}(\bar{\partial}_u)$ <u>is</u> the dimension.

Variation: For parametrized moduli spaces, one must add the
dimension of the parameter space. Eg. for Fukaya product μ^d,

$$\operatorname{vdim} \mathcal{M}_{\zeta^{d+1}}(\{y_0, y_1, \dots, y_n\}) = i(y_0) - \sum_{j=1}^{n} i(y_j) + d - 2$$

Therefore, when looking for zero dimensional components, we must have

$$i(y_0) - \sum_{j=1}^{n} i(y_j) + d-2 = 0$$

$$\underset{\text{degree of output}}{i(y_0)} = \underbrace{\sum_{j=1}^{n} i(y_j)}_{\text{sum of degrees of inputs}} + \underbrace{2-d}_{\text{shift}}$$

This is why, when everything is graded,

$$\mu^d : CF(L_{d-1}, L_d) \otimes \cdots \otimes CF(L_0, L_1) \longrightarrow CF(L_0, L_d)[2-d]$$

has degree $2-d$!