Lecture 24 Indices of graded Lagrangian intersection

Last couple of lectures: Lagrangian grassmannian, Maslov classes, graded Lagrangians.

We now seek to define a grading on $CF((L_0, \alpha_0^\#), (L_1, \alpha_1^\#))$, where $(L_i, \alpha_i^\#)$ are graded Lagrangians in (M, ω, η^2)

Need a bit more linear algebra of Lagrangian Grassmannian $Gr(V)$ where (V, ω_V) is a symplectic vector space.

<u>Crossing form of a pair of paths</u> Let (λ_0, λ_1) be two paths $[0,1] \to Gr(V)$
For generic $s \in [0,1]$ and generic paths, $\lambda_0(s) \cap \lambda_1(s) = \{0\}$.
But for some values of s, $\lambda_0(s)$ and $\lambda_1(s)$ fail to be transverse.
At such a point, we define a quadratic form on $\lambda_0(s) \cap \lambda_1(s)$ as follows:

Choose continuous families of linear maps $\phi_{k,r,s} : \lambda_k(s) \to \lambda_k(r)$ for $k = 0, 1$ and $|r - s|$ small such that $\phi_{k,s,s} = Id$

Then define
$$q_{\lambda_0,\lambda_1}(s)(v) = -\left(\frac{d}{dr}\right)_{r=s} \omega_V\left(\phi_{0,r,s}(v), \phi_{1,r,s}(v)\right)$$
for $v \in \lambda_0(s) \cap \lambda_1(s)$

It is independent of $\phi_{k,r,s}$.

For generic paths, $\lambda_0(s) \cap \lambda_1(s)$ is at most one dimensional, and $q_{\lambda_0,\lambda_1}(s)$ is non-zero when the dimension is one; it can be either positive or negative in this case.

Now let $\mathcal{P}^- Gr(V)$ be the space of paths $\lambda : [0,1] \to Gr(V)$ such that:

- $\lambda(0) \pitchfork \lambda(1)$
- the pair $(\lambda, \lambda(1))$, where the second component denotes the constant path at $\lambda(1)$, has negative definite crossing form at $s = 1$.

We associate an <u>index</u> $I(\lambda)$ to $\lambda \in \mathcal{P}^- Gr(V)$ as

$$I(\lambda) = \sum_{0 < s < 1} \operatorname{sign}(q_{\lambda, \lambda(1)}(s))$$

where $q_{\lambda,\lambda(1)}(s)$ is the crossing form on $\lambda(s) \cap \lambda(1)$.
(we can assume λ is generic in $\mathcal{P}^- Gr(V)$ so $\operatorname{sign}(q_{\lambda,\lambda(1)}(s)) = \pm 1$ at each crossing.)

<u>Graded Lagrangian Grassmannian</u>: We have $Gr(V) \cong U(n)/O(n) \xrightarrow{\det^2} S^1$

Let $Gr^\#(V)$ be the covering space defined via pullback of $\mathbb{R} \to S^1$

$$\begin{array}{ccc} Gr^\#(V) & \longrightarrow & \mathbb{R} \\ \pi \downarrow & & \downarrow \\ Gr(V) & \longrightarrow & S^1 \end{array}$$

Call this the Graded Lagrangian Grassmannian.

Let $\Lambda_0^\#, \Lambda_1^\# \in Gr^\#(V)$ be a pair of Graded Lagrangian subspaces
We associate an <u>absolute index</u> $i(\Lambda_0^\#, \Lambda_1^\#)$ to this pair:

$i(\Lambda_0^\#, \Lambda_1^\#) = I(\pi \circ \lambda^\#)$, where $\lambda^\# : [0,1] \to Gr^\#(V)$ is a path such that $\lambda^\#(0) = \Lambda_0^\#$, $\lambda^\#(1) = \Lambda_1^\#$, and $\pi \circ \lambda^\# : [0,1] \to Gr(V)$ lies in $\mathcal{P}^- Gr(V)$.

Now suppose that (M,ω) is a symplectic manifold and $\eta^2 : (\Lambda^{top}_{\mathbb{C}} TM)^{\otimes 2} \to \underline{\mathbb{C}}$ is a trivialization.

Let L_0, L_1 be two transversely intersecting Lagrangian submanifolds we get maps

$$\begin{tikzcd} L_k \arrow[r, "\tilde{j}"] \arrow[rr, bend right, "\alpha_{L_k}"'] & Gr(TM) \arrow[r, "\alpha"] & S^1 \end{tikzcd}$$

and we suppose $\mu_{L_k} = [\alpha_{L_k}] = 0$ in $H^1(L_k; \mathbb{Z})$.
We pick lifts $\alpha^\#_k : L_k \to S^1$, so $(L_k, \alpha^\#_k)$ is a graded Lagrangian submanifold.

The choice of trivialization η^2 allows us to construct a covering space

$$\begin{tikzcd} Gr^\#(TM) \arrow[r] \arrow[d] & \mathbb{R} \arrow[d] \\ Gr(TM) \arrow[r, "\alpha"] & S^1 \end{tikzcd}$$

which is a fiberwise $\mathbb{Z}$-fold cover of $Gr(TM)$.
Then the choice of grading $\alpha^\#_k$ on L_k amounts to a lift

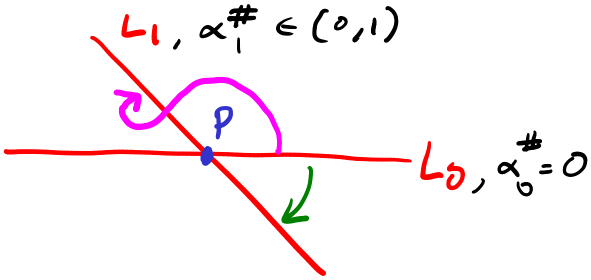
$$\begin{tikzcd} & Gr^\#(TM) \arrow[d] \\ & Gr(TM) \arrow[d] \\ L_k \arrow[uur, "j^\#_k"] \arrow[ur, "\tilde{j}_k"] \arrow[r, "j_k"] & M \end{tikzcd}$$

At a point $p \in L_0 \cap L_1$, we get a pair of graded Lagrangian subspaces

$$j^\#_0(p), j^\#_1(p) \in Gr^\#(T_pM)$$

and we define the index of p: $i(p) = i(j^\#_0(p), j^\#_1(p))$

Examples in dimension 2:



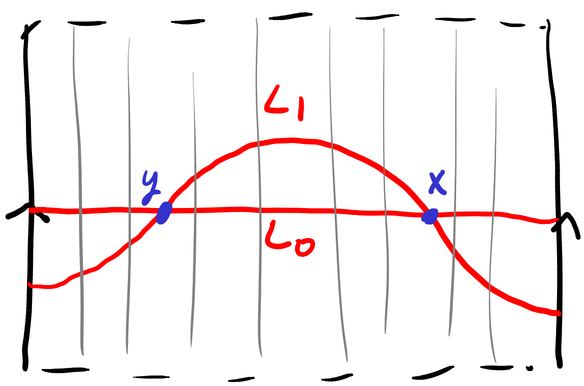
The short clockwise rotation from L_0 to L_1 lies in $\mathcal{P}^-Gr(T_pM)$
But it doesn't lift to $Gr^\#(T_pM)$ properly

The path that rotates L_0 counter-clockwise <u>past</u> L_1 and then back lies in $\mathcal{P}\,Gr(T_pM)$ and does lift properly. It has one positive crossing, so $i(p) = 1$

If we were to take $\alpha_1^\# \in (-1,0)$, we would have computed $i(p)=0$ instead.

on T^*S^1

Gray lines = line field



Can take
$\alpha_0^\# \equiv 0$,
$\alpha_1^\# \in (-1,1)$
then
$i(x) = 0$
$i(y) = 1$

<u>Next</u>: ∂ has degree 1: