

Lecture 23 Graded Lagrangians

Recall (V, ω) symplectic vector space $\rightsquigarrow \text{Gr}(V) = \{L \subset V\}$
Lagrangian Grassmannian.

Now let (M, ω) be a symplectic manifold

$$TM \rightsquigarrow \text{Gr}(TM) \cong \text{Gr}(T_p M)$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ M & M & \ni p \end{array}$$

Take the Lagrangian Grassmannian of each tangent space of M ; we get a fiber bundle over M with fiber $\text{Gr}(\mathbb{R}^{2n}) \cong U(n)/O(n)$, and structure group $U(n)$.

Recall the function $\det^2: U(n)/O(n) \rightarrow S^1$

We want to "globalize" this to a function $\alpha: \text{Gr}(TM) \rightarrow S^1$ whose restriction to each fiber is isomorphic to \det^2 :

$$\begin{array}{ccc} \text{Gr}(T_p M) & \xrightarrow{\alpha} & S^1 \\ \text{iso depends} & \nearrow \cong & \\ \text{on } p. & U(n)/O(n) & \xrightarrow{\det^2} \end{array}$$

This isn't automatically possible to do continuously since the isomorphism $\text{Gr}(T_p M) \rightarrow U(n)/O(n)$ is not canonical.

But now consider $K_M = \Lambda_{\mathbb{C}}^n T^*M$, the bundle of top forms, a complex line bundle.

Suppose $\eta^2 \in \Gamma(M, K_M^{\otimes 2})$ is a section that never vanishes.

Then we may define a map

$$\alpha: \text{Gr}(TM) \rightarrow S^1$$

for $L \subset T_p M$, let v_1, \dots, v_n be an \mathbb{R} -basis of L , and set

$$\alpha(L) = \frac{\eta^2(v_1 \wedge \dots \wedge v_n)}{|\eta^2(v_1 \wedge \dots \wedge v_n)|}$$

When restricted to a single fiber, this map is equivalent to $\det^2: U(n)/O(n) \rightarrow S^1$

The existence of a nowhere vanishing section $\eta^2 \in \Gamma(M, K_M^{\otimes 2})$ is equivalent to the triviality of the line bundle $K_M^{\otimes 2}$, which is equivalent to $c_1(K_M^{\otimes 2}) = 0$ since $c_1(K_M^{\otimes 2}) = 2c_1(K_M) = -2c_1(TM) = -2c_1(M)$. we find

Proposition There is a continuous map $\alpha: \text{Gr}(TM) \rightarrow S^1$ that is equivalent to \det^2 on each fiber if and only if $2c_1(M) = 0 \in H^2(M; \mathbb{Z})$

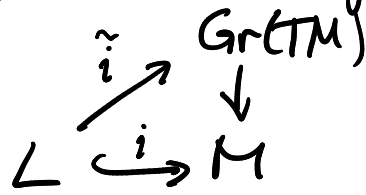
Remarks (1) It is possible that $2c_1(M) = 0$ but $c_1(M) \neq 0$ (c_1 is 2-torsion): Enriques surface.
 (2) The condition $2c_1(M) = 0$ is a kind of "Calabi-Yau condition".

When $2c_1(M) = 0$ and $\eta^2 \in \Gamma(M, K_M^{\otimes 2})$ nonvanishing exists, there can be several homotopy classes of sections η^2 ; the set of homotopy classes is a torsor over

$$H^1(M, \mathbb{Z}) \cong [M, S^1]$$

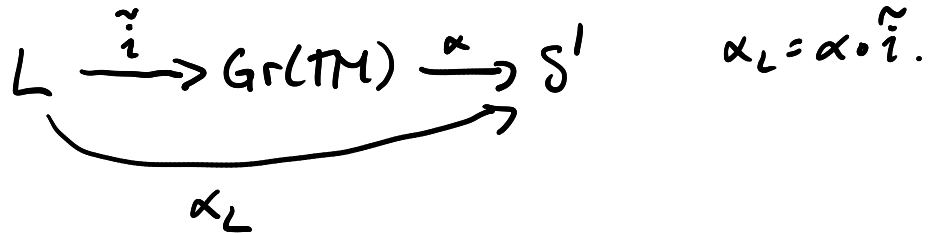
Now let (M, ω, η^2) be given, and take a Lagrangian submanifold $L \subset M$.

There is a natural lift



$$\tilde{i}(p) = T_p L \in \text{Gr}(T_p M)$$

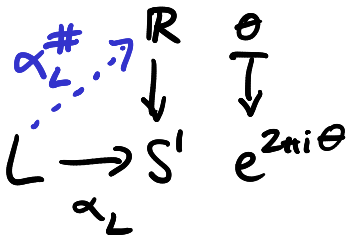
Thus we obtain a function



The homotopy class of this map $\mu_L = [\alpha_L] \in [L, S^1] \cong H^1(L, \mathbb{Z})$ is called the Maslov class of L .

Our interest will be in Lagrangians $L \subset M$ whose Maslov class vanishes.

$\mu_L = 0$ if and only if the map $\alpha_L : L \rightarrow S^1$ admits a lift to \mathbb{R}



When such a lift exists, the set of possible choices is a torsor over $\mathbb{Z} : \alpha_L^\# \rightarrow \alpha_L^\# + n$.

Def A graded Lagrangian submanifold in (M, ω, η^2) is a pair $(L, \alpha^\#)$ where $L \subset M$ is Lagrangian and $\alpha^\# : L \rightarrow \mathbb{R}$ is such that $\exp(2\pi i \alpha^\#) = \alpha_L$

Example of surfaces: If $\dim M = 2$, $K_M = T^*M$,
 and $K_M^{\otimes 2} = (T^*M)^{\otimes 2}$ so choosing η^2
 is equivalent to choosing an unoriented line field
 or a foliation of M .

If $L \subset M$ is a curve (automatically Lagrangian), then
 μ_L is proportional to the rotation # of L with
 respect to the line field.