Lecture 23 Graded Lagrangians

Recall (V,ω) symplectic vector space $\rightsquigarrow$ $Gr(V) = \{L \subset V\}$
Lagrangian Grassmannian.

Now let (M^{2n},ω) be a symplectic manifold

$$\begin{array}{ccccc} TM & \rightsquigarrow & Gr(TM) & \supseteq & Gr(T_pM) \\ \downarrow & & \downarrow & & \downarrow \\ M & & M & \ni & p \end{array}$$

Take the Lagrangian Grassmannian of each tangent space of M; we get a fiber bundle over M with fiber $Gr(\mathbb{R}^{2n}) \cong U(n)/O(n)$, and structure group $U(n)$.

Recall the function $\det^2 : U(n)/O(n) \longrightarrow S^1$
We want to "globalize" this to a function $\alpha : Gr(TM) \to S^1$ whose restriction to each fiber is isomorphic to $\det^2$:

$$\begin{tikzcd} Gr(T_pM) \arrow[r, "\alpha"] \arrow[d, phantom, "\cong"] & S^1 \\ U(n)/O(n) \arrow[ur, "\det^2"'] & \end{tikzcd}$$

iso depends on p.

This isn't automatically possible to do continuously since the isomorphism $Gr(T_pM) \to U(n)/O(n)$ is not canonical.

But Now consider $K_M = \Lambda^n_{\mathbb{C}} T^*M$, the bundle of top forms, a complex line bundle.
Suppose $\eta^2 \in \Gamma(M, K_M^{\otimes 2})$ is a section that never vanishes.

Then we may define a map

$$\alpha : Gr(TM) \longrightarrow S^1$$

for $L \subset T_pM$, let $v_1, \dots, v_n$ be an $\mathbb{R}$-basis of L, and set

$$\alpha(L) = \frac{\eta^2(v_1 \wedge \cdots \wedge v_n)}{|\eta^2(v_1 \wedge \cdots \wedge v_n)|}$$

when restricted to a single fiber, this map is equivalent to $\det^2 : U(n)/O(n) \to S^1$

The existence of a nowhere vanishing section $\eta^2 \in \Gamma(M, K_M^{\otimes 2})$ is equivalent to the triviality of the line bundle $K_M^{\otimes 2}$, which is equivalent to $c_1(K_M^{\otimes 2}) = 0$

since $c_1(K_M^{\otimes 2}) = 2c_1(K_M) = -2c_1(TM) = -2c_1(M)$.

we find

<u>Proposition</u> There is a continuous map $\alpha : Gr(TM) \to S^1$ that is equivalent to $\det^2$ on each fiber if and only if $2c_1(M) = 0 \in H^2(M; \mathbb{Z})$

<u>Remarks</u> (1) It is possible that $2c_1(M) = 0$ but $c_1(M) \neq 0$ (c_1 is 2-torsion) : Enriques surface.

(2) The condition $2c_1(M) = 0$ is a kind of "Calabi-Yau condition".

When $2c_1(M) = 0$ and $\eta^2 \in \Gamma(M, K_M^{\otimes 2})$ nonvanishing exists, there can be several homotopy classes of sections η^2; the set of homotopy classes is a torsor over

$$H^1(M, \mathbb{Z}) \cong [M, S^1]$$

Now let (M, ω, η^2) be given, and take a Lagrangian submanifold $L \subset M$.

There is a natural lift

$$\begin{tikzcd} & Gr(TM) \arrow[d] \\ L \arrow[ur, "\tilde{i}"] \arrow[r, hook, "i"] & M \end{tikzcd}$$

$$\tilde{i}(p) = T_pL \in Gr(T_pM)$$

Thus we obtain a function

$$\begin{tikzcd} L \arrow[r, "\tilde{i}"] \arrow[rr, bend right, "\alpha_L"'] & Gr(TM) \arrow[r, "\alpha"] & S^1 \end{tikzcd} \qquad \alpha_L = \alpha \circ \tilde{i}.$$

The homotopy class of this map $\mu_L = [\alpha_L] \in [L, S^1] \cong H^1(L, \mathbb{Z})$ is called the <u>Maslov class</u> of L.

Our interest will be in Lagrangians $L \subset M$ whose Maslov class vanishes.

$\mu_L = 0$ if and only if the map $\alpha_L : L \to S^1$ admits a lift to $\mathbb{R}$

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ L \arrow[ur, dashed, "\alpha_L^\#"] \arrow[r, "\alpha_L"'] & S^1 \end{tikzcd} \qquad \begin{tikzcd} \theta \arrow[d, mapsto] \\ e^{2\pi i \theta} \end{tikzcd}$$

When such a lift exists, the set of possible choices is a torsor over $\mathbb{Z}$: $\alpha_L^\# \to \alpha_L^\# + n$.

Def A graded Lagrangian submanifold in (M, w, η^2) is a pair $(L, \alpha^\#)$ where $L \subset M$ is Lagrangian and $\alpha^\# : L \to \mathbb{R}$ is such that $\exp(2\pi i \alpha^\#) = \alpha_L$

Example of surfaces: If $\dim M = 2$, $K_M = T^*M$, and $K_M^{\otimes 2} = (T^*M^{\otimes 2})$ so choosing η^2 is equivalent to choosing an <u>unoriented line field</u> or a foliation of M.

If $L \subset M$ is a curve (automatically Lagrangian), then μ_L is proportional to the rotation # of L with respect to the line field.