Lecture 22 Lagrangian Grassmannian

So far, we have constructed an "ungraded" A_∞-category. Introducing a grading is an important refinement since it constrains the algebra a lot, and also since in mirror symmetry we want to match the Fukaya category to a graded category.

Grassmannian: $G(k,n)$ = k-planes in n-space.

Over $\mathbb{R}$: $G_{\mathbb{R}}(k,n) \cong GL(n,\mathbb{R}) / P(k,n)$

where $P(k,n) = \left\{ \left(\begin{array}{c|c} A_{k\times k} & B \\ \hline 0 & C \end{array} \right) \right\}$ is the subgroup

that fixes $\mathbb{R}^k \times 0 \subseteq \mathbb{R}^n$

alternatively, picking an inner product, we can write

$$G_{\mathbb{R}}(k,n) \cong O(n) / (O(k) \times O(n-k))$$

Over $\mathbb{C}$: $G_{\mathbb{C}}(k,n) \cong GL(n,\mathbb{C}) / P_{\mathbb{C}}(k,n) \cong U(n)/(U(k)\times U(n-k))$

Now consider $V = \mathbb{R}^{2n} \ni (x_1, y_1, x_2, y_2, \dots, x_n, y_n)$ with symplectic form $\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$ (thought of as a bilinear form on V)

Define the Lagrangian Grassmannian $Gr(V) = \left\{ \begin{array}{c} \text{Lagrangian linear} \\ \text{subspaces of } V \end{array} \right\}$

We will see that $Gr(V) = U(n)/O(n)$:

consider $\mathbb{C}^n$ with the hermitian inner product

$$\langle z, w \rangle = \sum_{i=1}^{n} \bar{z}_i w_i$$

in terms of $z_i = x_i + y_i\sqrt{-1}$ $\quad w_i = x_i' + y_i'\sqrt{-1}$

$$\langle z, w \rangle = \sum_i \left[(x_i x_i' + y_i y_i') + (x_i y_i' - x_i' y_i)\sqrt{-1} \right]$$

$$= g(z,w) + \omega(z,w)\cdot\sqrt{-1}$$

where g is a positive definite symmetric form and ω is the symplectic form.

① Now if $(v_1, v_2, \dots, v_n)$ is a unitary $\mathbb{C}$-basis of $\mathbb{C}^n$, then

$\mathrm{Span}_{\mathbb{R}}(v_1, \dots, v_n)$ is a Lagrangian subspace.

② Conversely if L is a Lagrangian subspace, and $(v_1, \dots, v_n)$ is a g-orthonormal $\mathbb{R}$-basis of L, then $(v_1, \dots, v_n)$ is also a unitary $\mathbb{C}$-basis of $\mathbb{C}^n$.

Now ② implies there is a transitive action of $U(n)$ on $Gr(\mathbb{C}^n)$ and ① implies that the stabilizer of $\mathbb{R}^n \subset \mathbb{C}^n$ is $O(n)$.
So $Gr(\mathbb{C}^n) \cong U(n)/O(n)$.

Fundamental group: There is a homomorphism $\det : U(n) \to S^1$
consider $\det^2 : U(n) \to S^1$

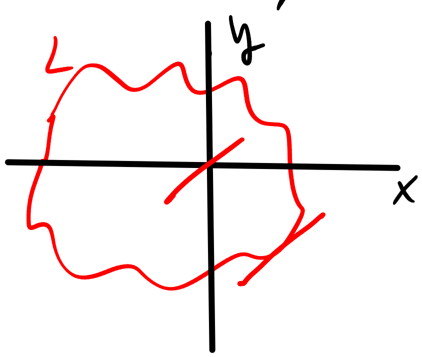
$$A \longmapsto (\det(A))^2$$

then since $\det(O(n)) = \{\pm 1\}$, we find $O(n) \subseteq \ker(\det^2)$
thus there is a well defined function.

$$Gr(V) \cong U(n)/O(n) \xrightarrow{\det^2} S^1$$

This map is an isomorphism on the fundamental group, and $\pi_1(Gr(V)) \cong \mathbb{Z}$.

Now consider a more nonlinear situation:
Take $(V = \mathbb{R}^{2n}, \omega = \sum dx_i \wedge dy_i)$ as a symplectic manifold, and consider Lagrangian submanifolds $L \subseteq V$

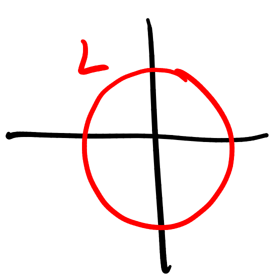


For a Lagrangian submanifold $L \subseteq V$ there is a map $\tau : L \to Gr(V)$

$$\tau(p) = \underbrace{T_pL \subseteq T_pV \cong V}_{\text{Lagrangian}}$$

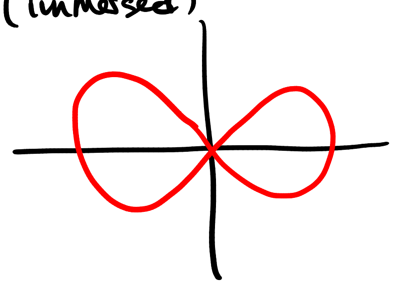
There is then a map $\varphi : L \xrightarrow{\tau} Gr(V) \xrightarrow{\det^2} S^1$
this φ is called the squared phase function of L.
since $S^1 \simeq K(\mathbb{Z},1)$. The homotopy class of φ corresponds to a cohomology class
$\mu \in H^1(L, \mathbb{Z})$
called the <u>Maslov class</u> of $L \subseteq V$.

<u>Example:</u>



$\mu = 2 \cdot (\text{top class})$

(immersed)



$\mu = 0$