Lecture 21 Fukaya's A_∞-category

So far, the operations in the Floer TFT are defined by counting maps of Riemann surfaces with fixed complex structure on the domain. We used parametrized moduli spaces (with variable complex structure) to prove relations between these maps.

The full Fukaya category is an A_∞-category whose higher operations are defined using parametrized moduli spaces.

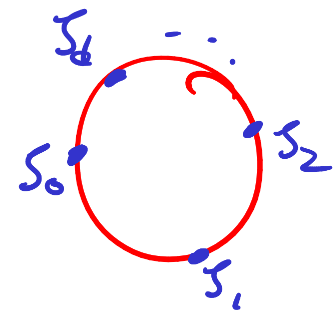
We continue to work over a field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K})=2$, and we assume that the Floer TFT for (M,ω) works as described in previous lectures:

* compactness, transversality, and gluing work as advertised
* $\partial^2=0$ on Floer cochain complexes $CF(L_0,L_1)$

These can be achieved by assuming (M,ω) is an "exact convex" symplectic manifold and restricting attention to exact Lagrangians.

Recall from the discussion of A_∞-categories and operads the moduli space
$$\mathcal{R}^{d+1} = \mathrm{Conf}_{d+1}(\partial D)/\mathrm{Aut}(D)$$
of disks with $d+1$ boundary marked points.

Each $r \in \mathcal{R}^{d+1}$ corresponds to a pointed-boundary Riemann surface $S_r = \hat{S}_r \setminus \Sigma_r$



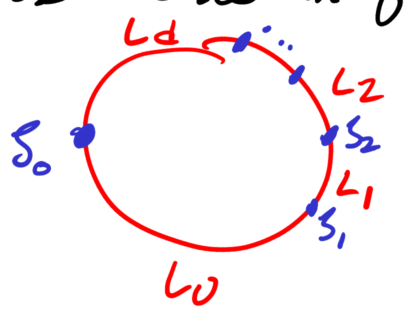
where $\Sigma = \{\zeta_0, \zeta_1, \dots, \zeta_d\}$ are the boundary marked points.

We set $\Sigma^- = \{\zeta_0\}$ $\Sigma^+ = \{\zeta_1, \dots, \zeta_d\}$

There is a universal family $\mathcal{S}^{d+1}$:

$$\begin{array}{ccc} S_r & \subset & \mathcal{S}^{d+1} \\ \downarrow & & \downarrow \pi \\ r & \in & \mathcal{R}^{d+1} \end{array}$$

Choose a collection of Lagrangians $(L_0, L_1, \dots, L_d)$



ζ_0 $(L_{\zeta_0,0}, L_{\zeta_0,1}) = (L_0, L_d)$

$(L_{\zeta_i,0}, L_{\zeta_i,1}) = (L_{i-1}, L_i)$ $1 \le i \le d$

Next, choose Floer data (H_i, J_i) for each pair $(L_0, L_d), (L_1, L_2), \dots, (L_{d-1}, L_d)$.

Next we need strip like ends: these should be chosen for the <u>whole family</u> $\mathcal{S}^{d+1} \xrightarrow{\pi} \mathcal{R}^{d+1}$. We call this a universal choice of strip like ends, and it is possible.

We must also choose perturbation data (K_r, J_r) for the family $\mathcal{S}^{d+1} \to \mathcal{R}^{d+1}$, compatible with the given Floer data on the strip like ends.

With these choices, we may, for a collection of chords

$$\{y_\zeta\} = \{y_{\zeta_0}, y_{\zeta_1}, \dots, y_{\zeta_d}\} \qquad y_{\zeta_0} \in \mathcal{C}(L_0, L_d), \quad y_{\zeta_i} \in \mathcal{C}(L_{i-1}, L_i)$$

consider the $\mathcal{R}^{d+1}$-parametrized moduli space.

$$\mathcal{M}_\zeta(\{y_\zeta\}) = \bigcup_{r \in \mathcal{R}^{d+1}} \mathcal{M}_{(S_r, k_r, J_r)}(\{y_\zeta\})$$

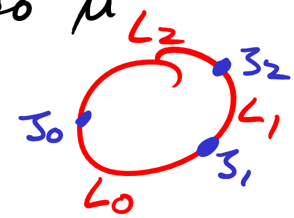
We can then count 0-dimensional components mod 2 to define

$$\mu^d : CF^{pr}(L_{d-1}, L_d) \otimes \cdots \otimes CF^{pr}(L_0, L_1) \to CF^{pr}(L_0, L_d)$$

for $d \geq 2$:

$$\mu^d(y_d, \dots, y_1) = \sum_{y_0} \# \mathcal{M}_\zeta(\{y_0, y_1, \dots, y_d\})^0 \, y_0$$

Note that, for $d=2$, $\mathcal{R}^{d+1} = \mathcal{R}^3 = pt$, so μ^2 just the TFT map for the surface



We also define $\mu^1 := \partial$.

Remark: the map μ^d is <u>not</u> the TFT map Φ_S where S the disk with $d+1$ marked boundary points. Φ_S counts 0-dimensional moduli spaces of maps with <u>fixed</u> complex structure on the domain.
On cohomology, Φ_S is iterated μ^2:

$$\Phi_S(a_d, \dots, a_1) = \mu^2(\cdots \mu^2(a_4, \mu^2(a_3, \mu^2(a_2, a_1)))\cdots).$$

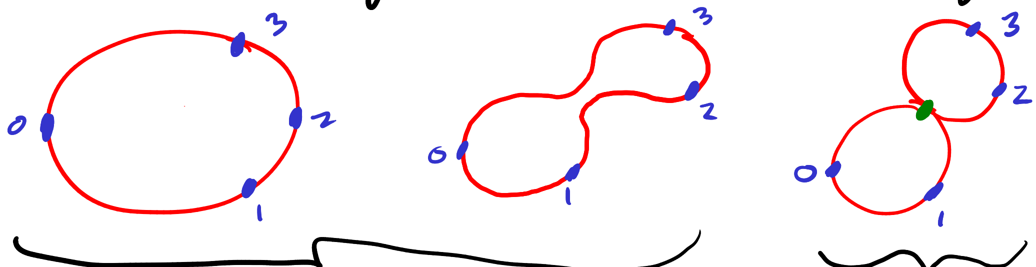
We would then like to prove the A_∞-equations, which mod 2 take the form

$$(\forall d \geq 1) \quad \sum_{e,i} \mu^{d-e+1}(a_d, \ldots, \mu^e(a_{i+e}, \ldots, a_{i+1}), a_i, \ldots, a_1) = 0$$

This comes by showing the left hand side counts boundary points in

$$\overline{\mathcal{M}}_{S^{d+1}}(\{y_S\})^1$$

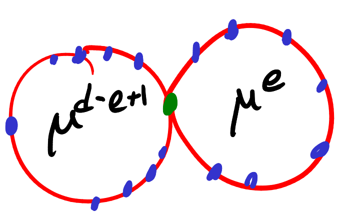
But there is actually an issue when a disk degenerates



here perturbation data were chosen when defining μ^3

here perturbation data were chosen when defining μ^2

Why should the limiting perturbation data for μ^3 be related to the perturbation data for μ^2? More generally



appears in boundary of $\overline{\mathcal{M}}_{S^{d+1}}$

We must impose a consistency condition that says the limiting pert. data for S^{d+1} is compatible with the pert. data for S^{e+1} $(e<d)$. This is called a consistent choice of universal perturbation data. It's existence is Lemma 9.5 in "Fukaya Cats and Picard Lefschetz theory".