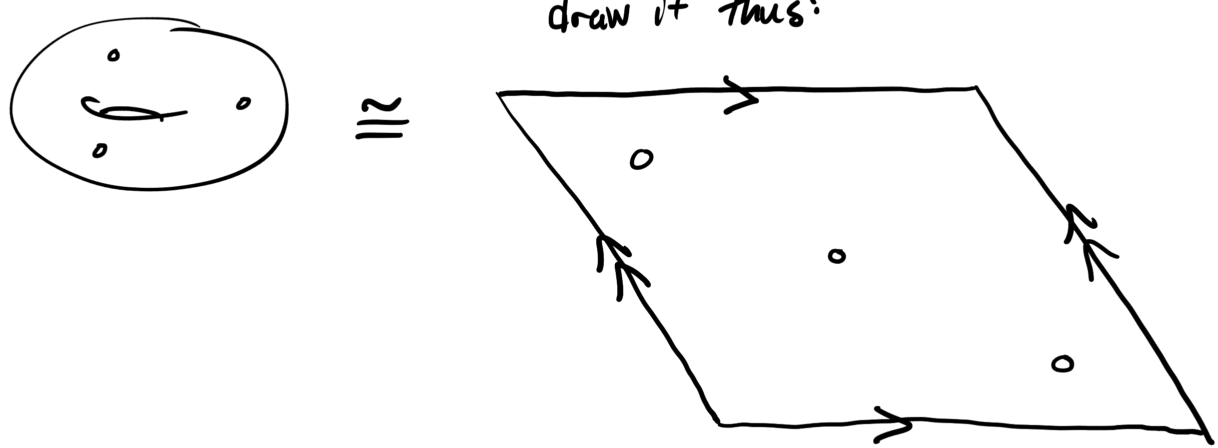
Lecture 20 Example: Mirror of $\mathbb{P}^2$

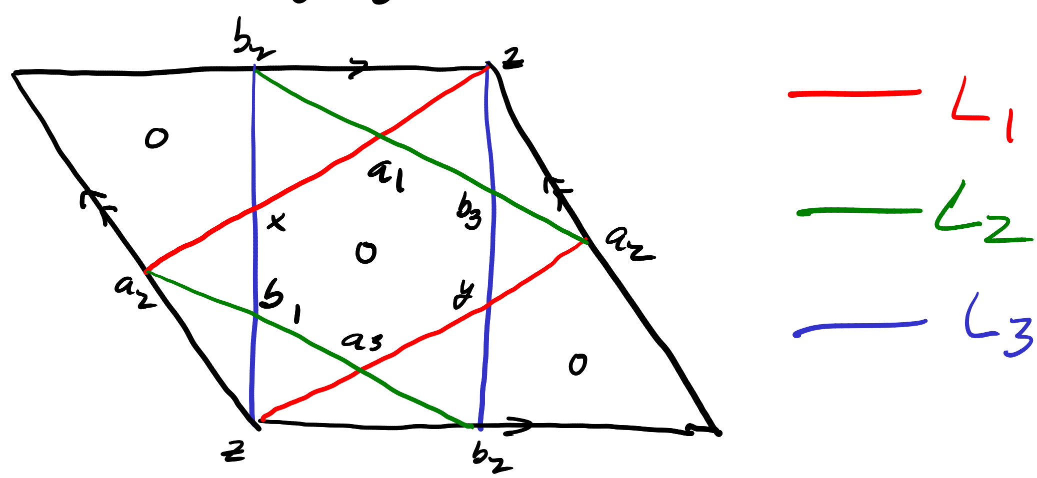
This is an example that both illustrates the Floer TFT an gives an instance of homological mirror symmetry.

Let M be a torus with 3 points removed.

draw it thus:



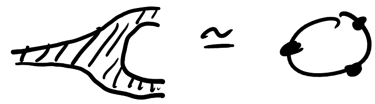
This lets us see 3 Lagrangian circles

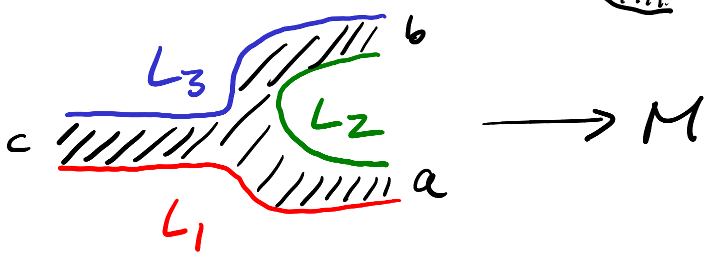


$$HF(L_1, L_2) = \langle a_1, a_2, a_3 \rangle$$
$$HF(L_2, L_3) = \langle b_1, b_2, b_3 \rangle$$
$$HF(L_1, L_3) = \langle x, y, z \rangle$$

There is a TFT map.

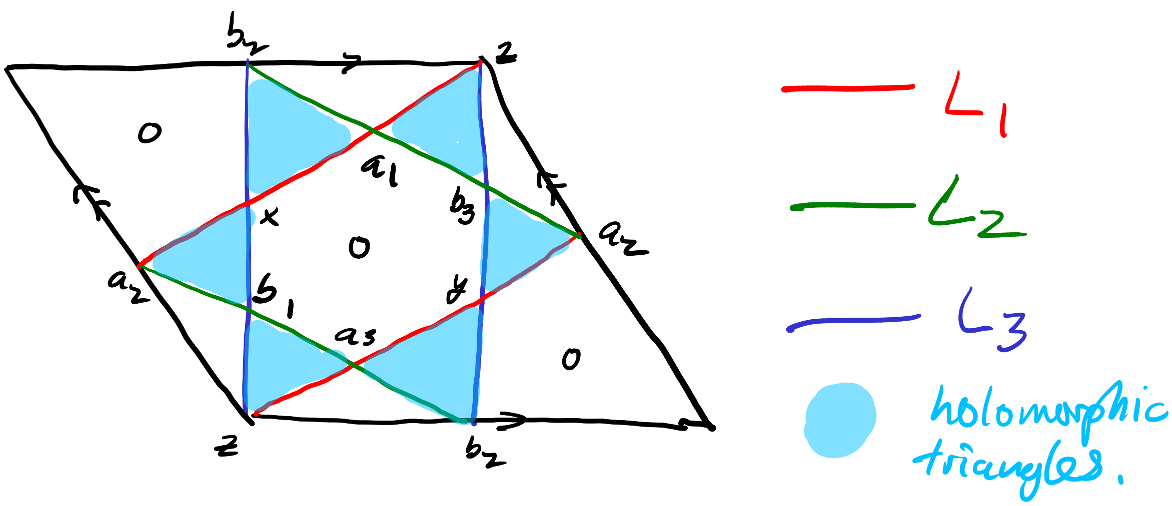
$$\mu^2 : HF(L_2, L_3) \otimes HF(L_1, L_2) \longrightarrow HF(L_1, L_3)$$

Associated to the surface 

counts 

Since $L_i \pitchfork L_j$, can use Floer data ($H \equiv 0$, J constant) and perturbation data ($K \equiv 0$, J constant).

Then we are really counting holomorphic maps to M. and we can use the Riemann mapping theorem to classify them.



Thus,

$$\mu^2(b_1, a_1) = 0 \qquad \mu^2(b_2, a_1) = x \qquad \mu^2(b_3, a_1) = z$$
$$\mu^2(b_1, a_2) = x \qquad \mu^2(b_2, a_2) = 0 \qquad \mu^2(b_3, a_2) = y$$
$$\mu^2(b_1, a_3) = z \qquad \mu^2(b_2, a_3) = y \qquad \mu^2(b_3, a_3) = 0$$

We can build a ring using μ^2:

(char $\mathbb{K} = 2$)

Define a $\mathbb{K}$-linear category $\mathcal{A}$ with objects L_1, L_2, L_3 as follows

$$\mathrm{Hom}_{\mathcal{A}}(L_i, L_j) = \begin{cases} 1_{L_i} & \text{if } i = j \\ HF(L_i, L_j) & i < j \\ 0 & i > j \end{cases}$$

where 1_{L_i} acts as an identity morphism.
and otherwise composition is given by μ^2.
Let

$$A = \bigoplus_{i,j} \mathrm{Hom}_{\mathcal{A}}(L_i, L_j) \quad \text{a } \mathbb{K}\text{-algebra}.$$

A can also be described as the path algebra of the quiver

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, bend left, "a_3"] \arrow[r, "a_2"] \arrow[r, bend right, "a_1"'] & \underset{2}{\bullet} \arrow[r, bend left, "b_3"] \arrow[r, "b_2"] \arrow[r, bend right, "b_1"'] & \underset{3}{\bullet} \end{tikzcd}$$

modulo the relations $\begin{cases} b_i a_i = 0 \\ b_j a_i = b_i a_j \end{cases}$

Now consider the triangulated category
$$D^b(A\text{-mod}_{fg})$$
the derived category of bounded complexes of finitely generated A-modules

A Theorem of A. Beilinson states

$$D^b(A\text{-mod}_{fg}) \cong D^b(\mathrm{Coh}\, \mathbb{P}^2_{\mathbb{K}}),$$

where $D^b(\mathrm{Coh}\, \mathbb{P}^2_{\mathbb{K}})$ is the bounded derived category of coherent sheaves on $\mathbb{P}^2$ over $\mathbb{K}$.

In fact, the category $\mathcal{A}$ with objects L_1, L_2, L_3 is isomorphic to the subcategory of $\mathrm{Coh}\,\mathbb{P}^2_{\mathbb{k}}$ with objects $\mathcal{O}$, $T \otimes \mathcal{O}(-1)$, $\mathcal{O}(1)$ where T is the tangent sheaf of $\mathbb{P}^2_{\mathbb{k}}$.