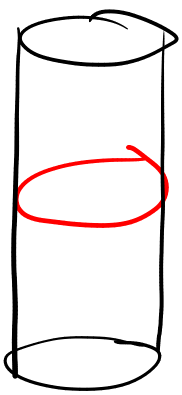
Lecture 19 Examples of Floer cohomology.

($\mathrm{Char}(\mathbb{K}) = 2$)

A source of computable examples comes from target surfaces, i.e. $\dim M = 2$ a symplectic form is just an area form, and any 1-dim'l submanifold $L \subset M$ is Lagrangian.

$M = T^*S^1 = \mathbb{R} \times S^1$: We can take $L = \{0\} \times S^1$

L



To compute $HF(L,L)$, we need to choose a perturbation datum. In particular, we need to choose a function $H : M \to \mathbb{R}$ such that $\varphi_H^1(L) \pitchfork L$.

Let (r, θ) be coords on $T^*S^1 = \mathbb{R} \times S^1$ such that $\omega = d\theta \wedge dr$ is the symplectic form.

Let $H(r,\theta) = h(\theta)$, where $h(\theta) : S^1 \to \mathbb{R}$ is a function with a unique nondegenerate ($h'' \neq 0$) maximum and a unique nondegenerate minimum. Then $dH = h'(\theta)\, d\theta$

and $\omega(-, X_H) = dH$ means

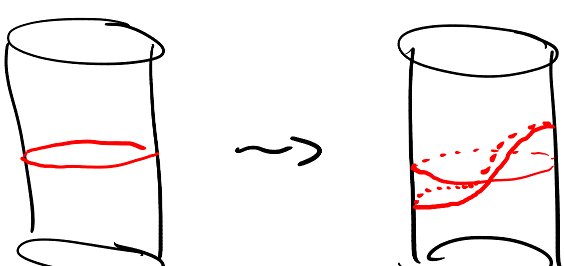
$$d\theta \cdot dr(X_H) - dr\, d\theta(X_H) = h'(\theta) d\theta$$

so $d\theta(X_H) = 0$ and $-dr(X_H) = h'(\theta)$

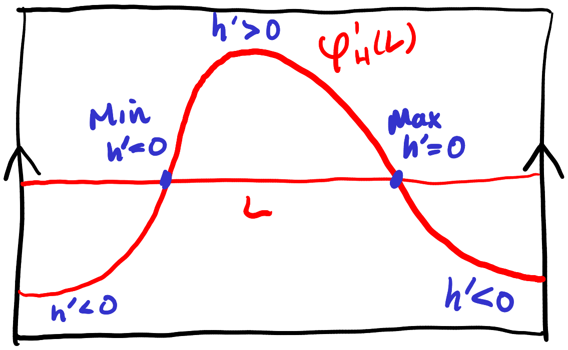
So $$X_H = h'(\theta) \frac{\partial}{\partial r}$$

and $$\varphi_H^1(r,\theta) = (r + h'(\theta), \theta)$$

so φ_H^1 shifts r-coordinate by an amount that depends on θ.

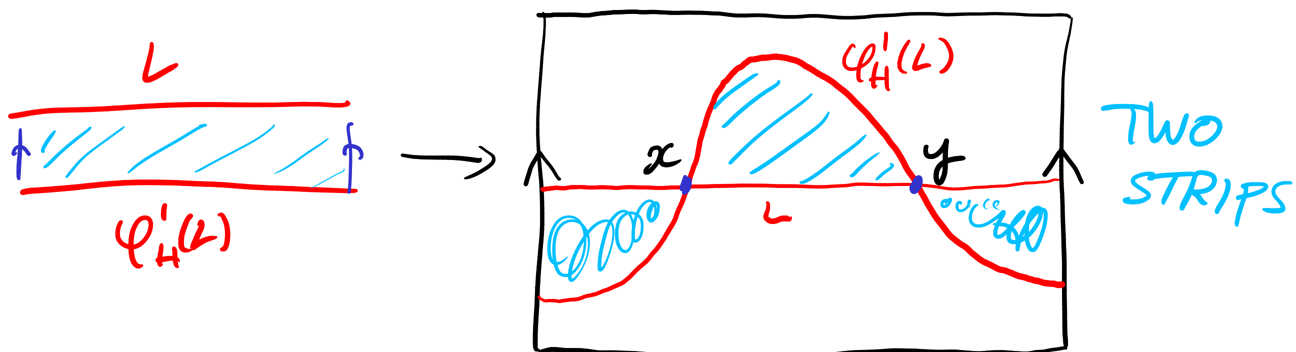


To make it a little more clear, we can unroll the cylinder



The generators of $CF(L,L)$ (for this perturbation) are the two intersection points of $\varphi_H^1(L)$ and L.

To find ∂, we must count strips joining these two points
these strips are visible in the perturbed picture



Since the bottom boundary corresponds to $\varphi_H^1(L)$, and the map is read right to left, there are two strips connecting x to y, so

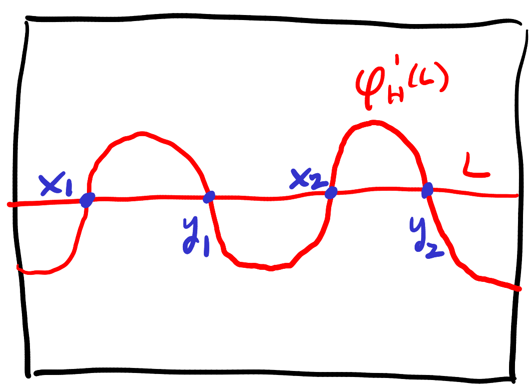
$$\partial(x) = 2y = 0 \quad (\mathrm{mod}\ 2)$$

and $\partial(y) = 0$ (no strips)

So $\partial = 0$ (and $\partial^2 = 0$)

The Floer cohomology is $HF(L,L) = CF(L,L) = \langle x, y \rangle$ is a rank 2 vector space over $\mathbb{K}$.

One could use different perturbations: eg



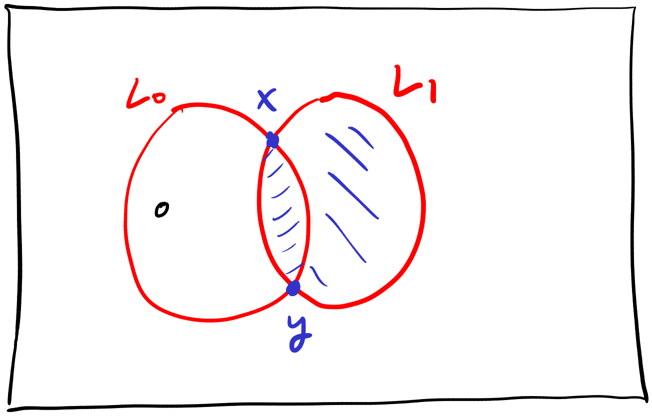
Then $\partial(x_1) = y_1 + y_2$
$\partial(x_2) = y_1 + y_2$
$\partial(y_1) = 0 = \partial(y_2)$

$\mathrm{Ker}(\partial) = \langle y_1, y_2, x_1 - x_2 \rangle$ rk = 3

$\mathrm{Im}(\partial) = \langle y_1 + y_2 \rangle$ rk = 1

$HF(L,L)$ rk = 2

An example where $\partial^2 \neq 0$: $M = \mathbb{R}^2 \setminus \{0\}$



$\partial(x) = y$

$\partial(y) = x$

$\partial^2 = Id$!

Issue: L_1 bounds a disk