Lecture 16 Gromov compactification

Recall we have $CF^{pr}(L_0, L_1) = \bigoplus_{y \in \mathcal{C}(L_0,L_1)} \mathbb{K} y$

$$\partial(y_+) = \sum_{y_-} \# \mathcal{M}_Z^*(y_-, y_+)\, y_-$$

and for a pointed boundary S with Lagrangian labels and perturbation data, a map

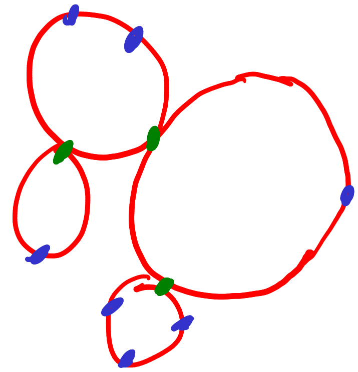
$$C\Phi_S : \bigotimes_{\zeta^+} CF^{pr}(L_{\zeta^+,0}, L_{\zeta^+,1}) \longrightarrow \bigotimes_{\zeta^-} CF^{pr}(L_{\zeta^-,0}, L_{\zeta^-,1})$$

$$C\Phi_S(\otimes y_{\zeta^+}) = \sum_{\{y_{\zeta^-}\}} \# \mathcal{M}_S(\{y_{\zeta^-}, y_{\zeta^+}\})(\otimes y_{\zeta^-})$$

where $\#$ means "count the isolated points modulo 2". At this point, we need a compactness result that implies these counts are finite. An after that, we need Prove relations such as $\partial \circ \partial = 0$ $\partial \circ C\Phi_S = C\Phi_S \circ \partial$ and we need to address the invariance and gluing.

It turns out that the same underlying idea is used for all of these things. It is a generalization for maps $u : S \to M$ of the Deligne-Mumford compactification $\overline{\mathcal{R}}^{d+1}$ of stable pointed disks.

Roughly, a <u>pseudo-holomorphic stable map</u> is a map from a nodal pointed-boundary Riemann surface to M:

u :  $\longrightarrow M$

* Each component of the domain carries Lagrangian labels and a perturbation datum. When two components meet at a node, there is a compatibility condition saying that the labels & pert. data match (i.e., they can be glued up)
* The map u is required to be pseudoholomorphic on each component, satisfy the boundary condition, and at each node the asymptotics from the two sides should match.
* The map is _stable_ in the sense that the set of holomorphic reparametrizations of the domain that leave the map u invariant is finite.

$\Rightarrow$ Each disk component on which u is constant has at least 3 special points (special points = marked points or nodes)

Given a sequence $\{u_n\}_{n=1}^{\infty}$ of stable pseudoholomorphic maps, there is a notion of convergence called _Gromov convergence_

which captures the process of node formation.

Thus given a moduli space $\mathcal{M}_S(\{y_s\})$, we can form $\overline{\mathcal{M}}_S(\{y_s\})$ by adding in all Gromov limits of sequences.

Next recall that there is a notion of energy for pseudo-holomorphic maps $E(u) \geq 0$.

<u>Gromov compactness</u>. For $C \geq 0$, the subset of $\overline{\mathcal{M}}_S(\{y_s\})$ where $E(u) \leq C$ is compact in the topology of Gromov convergence (every sequence has a convergent subsequence)

The point of using ω-compatible almost complex structures is that they allow us to obtain the energy bound $E(u) \leq C$ from the topology of the situation: maps in a fixed (relative) homology class have an a priori bound on their energy.

We impose the regularity hypothesis that says $\overline{\mathcal{M}}_S(\{y_s\})$ is a manifold with boundary and corners. (More on achieving this later).

Then Gromov compactness implies that the collection of zero dimensional components of $\overline{\mathcal{M}}_S(\{y_s\})$ is a finite set, and we can count its points (mod 2).

To prove relations such as $\partial \circ \partial = 0$ and $\partial \circ C\Phi_S = C\Phi_S \circ \partial$, we use the 1-dimensional components of $\overline{\mathcal{M}}_S(\{y\})$ and the fact that a compact 1-manifold has an even # of boundary points.