Lecture 15 The Lagrangian Floer TFT, II

Recall: (M,ω) symplectic $\mathcal{J} = \{\omega\text{-compatible a.c.s.}\}$

$S = \hat{S} \setminus (\Sigma^- \sqcup \Sigma^+)$ pointed boundary Riemann surface with strip like ends $\varepsilon_\zeta : \mathbb{R}^\pm \times [0,1] \to S$ for $\zeta \in \Sigma^\pm$.

A set of Lagrangian labels $\{L_C\}_{C \in \pi_0(\partial S)}$

Floer data $\left(H_\zeta \in C^\infty([0,1], \mathcal{H}),\ J_\zeta \in C^\infty([0,1], \mathcal{J}) \right)$ for each pair $(L_{\zeta,0}, L_{\zeta,1})$.

compatible perturbation data $\left(K \in \Omega^1(S, \mathcal{H}),\ J \in C^\infty(S, \mathcal{J}) \right)$

$Y = \text{Ham.v.f.}(K) \in \Omega^1(S, C^\infty(TM))$

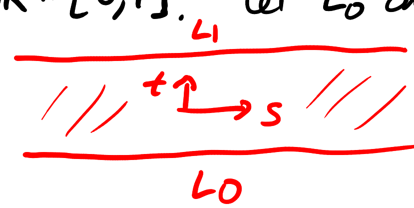
$\mathcal{C}(L_{\zeta,0}, L_{\zeta,1}) = \{ y : [0,1] \to M \mid y(i) \in L_{\zeta,i}\ (i=0,1),\ \dot{y}(t) = X_{H_\zeta}(t, y(t)) \}$

Moduli space $\mathcal{M}(\{y_\zeta\}_{\zeta \in \Sigma})$ = solutions $u : S \to M$ of

$$\begin{cases} (du - Y)^{0,1} = 0 \\ u(C) \subset L_C \quad \text{for } C \in \pi_0(\partial S) \\ \lim_{s \to \pm\infty} u(\varepsilon_\zeta(s,t)) = y_\zeta(t) \quad \text{for } \zeta \in \Sigma^\pm \end{cases}$$

These moduli spaces are used to define the TFT operation.

First, consider the case $S = Z = \mathbb{R} \times [0,1]$. Let L_0 and L_1 be the Lagrangian labels.



Choose Floer datum (H, J) for (L_0, L_1). Then take an S-TRANSLATION INVARIANT Perturbation datum (K, J')

$$K = H(t)\,dt \qquad J'(s,t) = J(t).$$

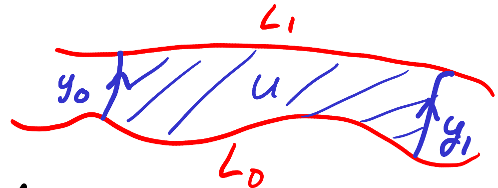
The equation $(du - Y)^{0,1} = 0$ is more explicitly written

$$\partial_s u + J(t,u)(\partial_t u - X(t,u)) = 0$$
$$u(s,0) \in L_0 \qquad u(s,1) \in L_1$$

The form of the equation is unchanged by translation in the s-direction. Write $\mathcal{M}_Z(y_0, y_1)$ for the set of solutions satisfying the asymptotic conditions

$$\lim_{s\to+\infty} u(s,t) = y_1(t) \qquad \lim_{s\to-\infty} u(s,t) = y_0(t),$$

where $y_0, y_1 \in \mathcal{C}(L_0, L_1)$



Then $\mathcal{M}_Z(y_0,y_1)$ has an $\mathbb{R}$-action, and we denote by $\mathcal{M}_Z^*(y_0,y_1) = \mathcal{M}_Z(y_0,y_1)/\mathbb{R}$

Let $\mathbb{K}$ be a field of $\mathrm{char}(\mathbb{K}) = 2$.
Define the Floer cochain group $CF^{pr}(L_0,L_1)$ to be a $\mathbb{K}$-vector space with a basis vector for each $y \in \mathcal{C}(L_0,L_1)$

$$CF^{pr}(L_0,L_1) := \bigoplus_{y \in \mathcal{C}(L_0,L_1)} \mathbb{K}\cdot y$$

The differential is defined in terms of the basis as

$$\partial(y_1) = \sum_{y_0} \# \mathcal{M}_Z^*(y_0,y_1)\, y_0$$

where $\#$ means "count the isolated points modulo 2".

More generally, for any pointed boundary R.S. $S = \hat{S} \setminus (\Sigma^- \amalg \Sigma^+)$

$$\mathcal{M}_S(\{y_\zeta\}_{\zeta\in\Sigma}) = \left\{ \begin{array}{l} (du - Y)^{0,1} = 0 \\ u(C) \subset L_C \\ \lim\limits_{s\to\pm\infty} u(\varepsilon_\zeta(s,t)) = y_\zeta(t) \end{array} \right\}$$

Define $C\Phi_S : \bigotimes_{\zeta^+\in\Sigma^+} CF^{pr}(L_{\zeta^+,0}, L_{\zeta^+,1}) \longrightarrow \bigotimes_{\zeta^-\in\Sigma^-} CF^{pr}(L_{\zeta^-,0}, L_{\zeta^-,1})$

in terms of basis: $C\Phi_S(\bigotimes_{\zeta^+} y_{\zeta^+}) = \sum_{\{y_{\zeta^-}\}} \# \mathcal{M}_S(\{y_{\zeta^-}, y_{\zeta^+}\})(\bigotimes_{\zeta^-} y_{\zeta^-})$

Now there are some things we would like to be able to claim

<u>Desiderata</u>: (I) on $CF^{pr}(L_0,L_1)$, $\partial\circ\partial=0$, so ∂ really is a differential, and $CF^{pr}(L_0,L_1)$ is a complex and we can take its cohomology $HF^{pr}(L_0,L_1)$

(II) $C\Phi_S$ is a chain map, so it induces a map

$$\Phi_S : \bigotimes_{\zeta^+} HF^{pr}(L_{\zeta^+,0}, L_{\zeta^+,1}) \to \bigotimes_{\zeta^-} HF^{pr}(L_{\zeta^-,0}, L_{\zeta^-,1})$$

(III) $HF^{pr}(L_0,L_1)$ is invariant under change of Floer data

(IV) Φ_S is invariant under change of perturbation data, and change of conformal structure on S.

(V) Φ_S satisfies the gluing axiom of a TFT.

These points are all somewhat intertwined, and in order to address them we first need to understand better how moduli spaces like $\mathcal{M}_S(\{y_\zeta\})$ are actually constructed.
Furthermore, simple examples show these desiderata cannot always be obtained without some restrictions on (M,ω) and/or L_C, we will clarify this as we go.

<u>Outline</u> of construction of $\mathcal{M}_S(\{y_\zeta\})$ as a manifold (not just a set)

Let $\mathcal{B}^\infty_S \subset \mathrm{Map}(S,M)$ be the space of smooth maps $u: S\to M$ such that $u(C)\subset L_C$ and which converge to some collection $\{y_\zeta\}$ on the striplike ends.
This is an ∞-dimensional "manifold" and its tangent space at u

$$C^\infty((S,\partial S),(u^*TM, u^*TL_C)) \subset C^\infty(S, u^*TM)$$

consisting of sections ξ of u^*TM over S such that, along the component $C\subset\partial S$, $\xi\in u^*TL_C$.

Over $\mathcal{B}_S^\infty$ there is a natural ∞-rank vector bundle $\mathcal{E}_S^\infty$ whose fiber at $u \in \mathcal{B}_S^\infty$ is $C^\infty(S, \Omega_S^{0,1} \otimes u^*TM)$

The mapping $\bar\partial_S : u \longmapsto (du - Y)^{0,1}$ can then be regarded as a section:

$$\begin{array}{c} \mathcal{E}_S^\infty \\ \downarrow \pi \\ \mathcal{B}_S^\infty \end{array} \bar\partial_S$$

and $\mathcal{M}(\{y_S\})$ is (a component of) vanishing set of this section.

To give $\mathcal{M}(\{y_S\})$ a manifold structure, we need to be able to formulate a condition of transversality in this ∞-dimensional setting, and require $\bar\partial_S$ to be transverse to the ~~zero~~ section.