# Lecture 14 The Lagrangian Floer TFT

<u>Worldsheets</u>: Let $\hat{S}$ be a compact Riemann surface w/ boundary $\Sigma \subset \partial\hat{S}$ a finite set partitioned as $\Sigma = \Sigma^- \amalg \Sigma^+$

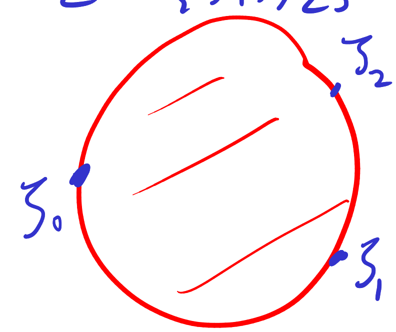
$S := \hat{S} \setminus \Sigma$ is a <u>pointed-boundary RS</u>.

$\Sigma^-$ = incoming boundary points

$\Sigma^+$ = outgoing boundary points.

$\Sigma^- = \{\zeta_0\}$

$\Sigma^+ = \{\zeta_1, \zeta_2\}$



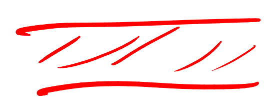

Eg $D$ = closed unit disk in $\mathbb{C}$

$H$ = upper half plane with point at $\infty$ incoming $\quad (\hat{H} = D)$

$\bar{H}$ = " with point at $\infty$ outgoing

$Z = \mathbb{R} \times [0,1]$ with coords $(s,t)$

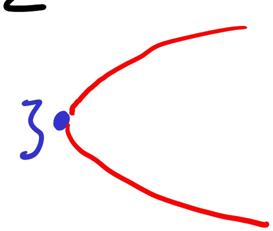
"$s = -\infty$" is incoming "$s = \infty$" outgoing



Fix $(M, \omega)$ a symplectic manifold

A <u>set of Lagrangian labels</u> for $S$ is a family $\{L_C\}$ of Lagrangian submanifolds $L_C \subset M$ indexed by connected components $C \subset \partial S = \partial\hat{S} \setminus \Sigma$

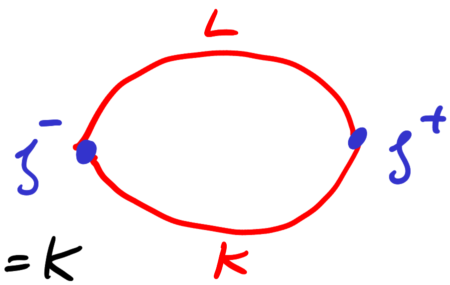
Now each $\zeta \in \Sigma \subset \hat{S}$ lies in the closure of exactly two components $C_{\zeta,0}, C_{\zeta,1}$




Hence there is a pair of Lagrangians $L_{\zeta,0} = L_{C_{\zeta,0}}$, $L_{\zeta,1} = L_{C_{\zeta,1}}$

<u>Convention</u>: For $\zeta \in \Sigma^-$ $\quad C_{\zeta,1}$ comes before $\zeta$ in natural boundary orientation,

For $\zeta \in \Sigma^+$ $\quad C_{\zeta,0}$ comes before $\zeta$ in " " orient.

Eg for $Z$:  $\sim$ 


$$L_{\zeta^-,0} = K \qquad L_{\zeta^+,0} = K$$
$$L_{\zeta^-,1} = L \qquad L_{\zeta^+,1} = L$$

For each pair $L_0, L_1$ of Lagrangians, we aim to construct a vector space $HF^{pr}(L_0,L_1)$ over a field $\mathbb{K}$, $\mathrm{char}(\mathbb{K})=2$ and for any point boundary R.S. $S = \hat{S} \setminus (\Sigma^- \sqcup \Sigma^+)$ a TFT map

$$\Phi_S : \bigotimes_{\zeta^+ \in \Sigma^+} HF^{pr}(L_{\zeta^+,0}, L_{\zeta^+,1}) \to \bigotimes_{\zeta^- \in \Sigma^-} HF^{pr}(L_{\zeta^-,0}, L_{\zeta^-,1})$$

Note that the inputs correspond to <u>outgoing</u> $\zeta$'s.

Satisfying gluing axiom.

To define this, we need a bit more structure on our domains. Let $S = \hat{S} \setminus (\Sigma^- \sqcup \Sigma^+)$ be a pointed-$\partial$ R.S. Denote by $Z^+ = \mathbb{R}^+ \times [0,1]$, $Z^- = \mathbb{R}^- \times [0,1]$ semi-infinite strips. A set <u>of strip-like ends</u> consists of proper holomorphic embeddings $\varepsilon_\zeta : Z^\pm \to S$ for $\zeta \in \Sigma^\pm$ such that $\varepsilon_\zeta^{-1}(\partial S) = \mathbb{R}^\pm \times \{0,1\}$, $\lim_{s \to \pm\infty} \varepsilon_\zeta(s,\cdot) = \zeta$.

This gives us distinguished coordinates $(s,t) \in \mathbb{R}^\pm \times [0,1]$ near each boundary puncture, which is useful for several reasons, including making the gluing procedure more precise.

We next introduce some perturbations that we will use in setting up the theory. This is the beginning of how we address the "virtual aspect" of intersection theory on the moduli space of pseudo-holomorphic maps.

Denote by $\mathcal{J}$ the space of $\omega$-compatible a.c.s. $J$ on $M$. For $T$ a manifold, denote by $C^\infty(T,\mathcal{J})$ the space of smooth $T$-parametrized families $\{J(t)\}_{t\in T}$

Let $\mathcal{H} = C^\infty(M,\mathbb{R})$ denote the space of smooth functions. Let $L_0, L_1 \subset M$ be a pair of Lagrangian submanifolds

A <u>Floer datum for $(L_0, L_1)$</u> consists of

$$H \in C^\infty([0,1],\mathcal{H}) \quad \text{and} \quad J \in C^\infty([0,1],\mathcal{J})$$

such that if $X$ is the time dependent Hamiltonian vector field of $H$: $\omega(-, X(t)) = dH(t)$ and $\varphi^1 : M \to M$ is its flow, then $\phi^1(L_0)$ intersects $L_1$ transversally.

Let $S$ be a pointed-$\partial$ R.S. with Lagrangian labels. and strip-like ends, and also suppose a Floer datum $(H_\zeta, J_\zeta)$ has been chosen for each of the pairs of submanifolds $(L_{\zeta,0}, L_{\zeta,1})$.

Then, a <u>perturbation datum for $S$</u> is a pair $(K, J)$ where $K \in \Omega^1(S,\mathcal{H})$ such that

$$K(\xi)|_{L_C} = 0 \quad \text{for all } \xi \in TC \subset T(\partial S)$$

and $J \in C^\infty(S,\mathcal{J})$. These must match with the chosen Floer data on the ends:

$$\varepsilon_\zeta^* K = H_\zeta(t)\,dt \qquad J(\varepsilon_\zeta(s,t)) = J_\zeta(t).$$

We use $K$ to introduce an inhomogeneous term into our pseudo-holomorphic map equation.

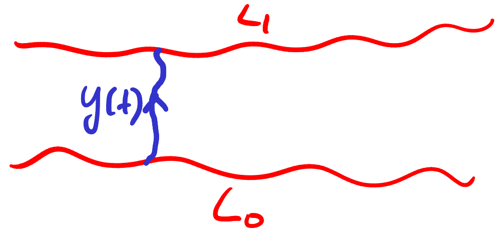
Now suppose, $S$, $\{L_C\}$, $(H_\zeta, J_\zeta)$, $(K, J)$ have been chosen satisfying all conditions above.

For each $\zeta$, denote by $\mathcal{C}(L_{\zeta,0}, L_{\zeta,1})$ the set of "chords" $y : [0,1] \to M$ such that

$y(0) \in L_{\zeta,0}$

$y(1) \in L_{\zeta,1}$

$\dot{y}(t) = X(t, y(t))$




Then $\mathcal{C}(L_{\zeta,0}, L_{\zeta,1})$ is in bijection with $\phi^1(L_{\zeta,0}) \cap L_{\zeta,1}$.

$K$ determines a vector field $Y \in \Omega^1(S, C^\infty(TM))$

For $\xi \in TS$ let $Y(\xi)$ be the Hamiltonian vector field of $K(\xi)$

The inhomogeneous pseudo-holomorphic map equation for $u : S \to M$ is

$$\circledast \quad \begin{cases} (du - Y)^{0,1} = 0 \\ u(C) \subset L_C \quad \text{for all components } C \subset \partial S \end{cases}$$

One can show that "finite energy solutions" converge to elements of $\mathcal{C}(L_{\zeta,0}, L_{\zeta,1})$ on the ends. So it makes sense to take a tuple $\{y_\zeta \in \mathcal{C}(L_{\zeta,0}, L_{\zeta,1})\}_{\zeta \in \Sigma}$

and consider the set $\mathcal{M}(\{y_\zeta\}_{\zeta \in \Sigma})$ of solutions to $\circledast$ with the asymptotic conditions

$$\lim_{s \to \pm\infty} u(\varepsilon_\zeta(s, \cdot)) = y_\zeta \quad \text{for } \zeta \in \Sigma^\pm$$

This is the moduli space we use to define $\Phi_S$