Lecture 13 Almost complex structures and Pseudo-holomorphic curves.

Let (M,ω) be a sympleetic manifold : $\omega \in \Omega^2(M)$, $d\omega = 0$, $\omega^n > 0$.
We can think of ω as a skew-symmetric 2-form on TM.
$\omega : TM^{\otimes 2} \to \mathbb{R}$.
An <u>almost complex structure</u> J on M is a section of $End(TM)$, i.e. a bundle map $J : TM \to TM$ such that $J^2 = -Id$.
a choice of an almost complex structure makes each tangent space T_pM into a $\mathbb{C}$-vector space with $(a+bi).v = av + bJ(v)$

An almost complex structure J is called <u>compatible with ω</u> if $g(u,v) = \omega(u, J(v))$ is a positive definite symmetric bilinear form, i.e., a Riemannian metric.

<u>Proposition</u> The space of almost complex structures compatible with a given symplectic form is contractible.

There is a fairly constructive proof of this fact due to B. Sévennec
<u>Lemma</u> Let $(\mathbb{C}^n, J_0 = i, g, \omega)$ denote the standard complex space with hermitian metric $\langle u,v \rangle = g(u,v) + i\omega(u,v)$
$$\langle (z_i),(w_i) \rangle = \sum_{i=1}^n z_i \overline{w_i}$$
Let $\mathcal{J}_c = \{ J \in Mat_{2n\times 2n} \mid J^2 = -Id, \ J \text{ compatible with } \omega \}$

Then, the "Cayley transform" $J \mapsto S := \frac{J - J_0}{J + J_0}$
induces a diffeomorphism
$$\mathcal{J}_c \longrightarrow \{ S \in Mat_{2n\times 2n} \mid \|S\|_g < 1, \ J_0 S + S J_0 = 0, \ S^T = S \}$$

Proof is a (somewhat lengthy) linear algebra exercise.

Now observe that the conditions $J_0 S + S J_0 = 0$ and $S^T = S$ are linear, while the condition $\|S\|_g < 1$ is convex. This shows that $\mathcal{J}_c$ is contractible.

Now given (M,ω), let $\mathbb{J}_c(\omega) \longrightarrow M$ be the fiber-bundle whose fiber at $p \in M$ is the space of linear maps $J_p : T_pM \to T_pM$, $J^2 = -\mathrm{Id}$ which are compatible with $\omega_p : T_pM^{\otimes 2} \to \mathbb{R}$. Then by the lemma, $\mathbb{J}_c(\omega) \to M$ has contractible fibers. The space $\mathcal{J}_c(\omega)$ of compatible a.c.s. on M is the space of sections of $\mathbb{J}_c(\omega)$, so it is also contractible. ∎

Rmk Why "almost"? On an almost complex manifold, (M,J), there need not exist local holomorphic coordinates. If such do exist, the a.c.s. is called integrable. Cf. Newlander–Nirenberg

Now let (Σ, j) be a Riemann surface $\dim_{\mathbb{R}} \Sigma = 2$, j an acs on Σ. There is a good theory of pseudo-holomorphic maps $u : (\Sigma, j) \to (M, J)$, and a good moduli theory when J is compatible with ω.

Consider a smooth map $u : \Sigma \to M$. for $p \in \Sigma$, $d_p u : T_p\Sigma \to T_{u(p)}M$ is a linear map. Now $(T_p\Sigma, j_p)$ and $(T_{u(p)}M, J_p)$ are complex vector spaces, but $d_p u$ is only $\mathbb{R}$-linear in general. We can take the $\mathbb{C}$-linear and $\mathbb{C}$-anti linear components.

$$\partial u = (du)^{1,0} = \tfrac{1}{2}(du - J \circ du \circ j) \qquad \partial u \circ j = J \circ \partial u$$

$$\bar\partial u = (du)^{0,1} = \tfrac{1}{2}(du + J \circ du \circ j) \qquad \bar\partial u \circ j = -J \circ \bar\partial u$$

u is pseudoholomorphic if $(du)^{0,1} = 0$.

The <u>energy</u> of a map is $E(u) = \frac{1}{2}\int_\Sigma |du|^2 \, dvol_\Sigma$

where $|du|^2$ is calculated using the metric $g(u,v) = \omega(u, Jv)$
It may appear to depend on a metric on Σ as well, but it is actually conformally invariant, so it only depends on the complex structure of Σ:
Let h be any metric compatible with the complex structure of Σ: means $h(v, jv) = 0$ for all v.
Any other metric with this property is $h' = e^{2\varphi} h$ for some function $\varphi : \Sigma \to \mathbb{R}$.
Then $|du|^2_{g,h'} = e^{-2\varphi} |du|^2_{g,h}$ and $dvol_{h'} = e^{2\varphi} dvol_h$

so $|du|^2_{g,h'} \, dvol_{h'} = |du|^2_{g,h} \, dvol_h$

<u>Prop</u>: If $u : \Sigma \to M$ is pseudo-holomorphic, and J is compatible with ω, then $E(u) = \int_\Sigma u^*\omega$

<u>Proof</u> Choose a local holomorphic coordinate $z = s + it$ such that $h(\partial_s, \partial_s) = 1$. Then $h(\partial_t, \partial_t) = 1$, $h(\partial_s, \partial_t) = 0$.

$$g(du(\partial_t), du(\partial_t)) = g(du(j\partial_s), du(j\partial_s)) = g(J\,du(\partial_s), J\,du(\partial_s))$$
$$= g(du(\partial_s), du(\partial_s))$$

And $g(du(\partial_s), du(\partial_t)) = g(du(\partial_s), J\,du(\partial_s)) = 0$

If ξ is an h-unit vector on Σ, $\xi = \cos\theta\,\partial_s + \sin\theta\,\partial_t$

Then $g(du(\xi), du(\xi)) = \cos^2\theta\, g(du(\partial_s), du(\partial_s)) + \sin^2\theta\, g(du(\partial_t), du(\partial_t))$
$= g(du(\partial_s), du(\partial_s))$.

$$|du|^2_{g,h} = \max_{h(\xi,\xi)=1} g(du(\xi), du(\xi)) = g(du(\partial_s), du(\partial_s)) = \omega(du(\partial_s), J\,du(\partial_s))$$
$$= \omega(du(\partial_s), du(j\partial_s)) = \omega(du(\partial_s), du(\partial_t))$$

so $|du|^2_{g,h} \, dvol_h = \omega(du(\partial_s), du(\partial_t)) \, ds \wedge dt = u^*\omega$.