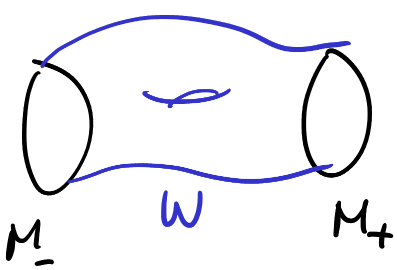
Lecture 11 Closed and open TFT

We will see that the discussion of $\bar{\mathcal{R}}^{d+1}$ and A_∞-structures fits nicely with ideas about boundary conditions in open string theory.

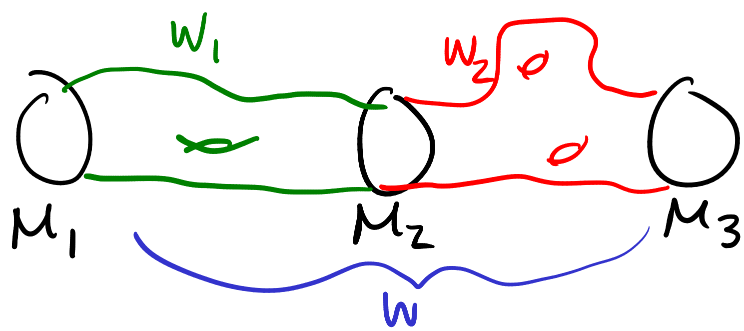
We begin by recalling the famous result

2D TFTs = commutative Frobenius algebras.

Def A $d+1$ dimensional topological field theory Z associates to each closed d-manifold M a $\mathbb{C}$-vectorspace $Z(M)$ and to each $(d+1)$-cobordism W, $\partial W = M_+ \amalg (-M_-)$ a map $Z(W) : Z(M_-) \to Z(M_+)$.



We require $Z(M_1 \amalg M_2) = Z(M_1) \otimes Z(M_2)$ and similarly for disjoint unions of cobordisms, and we require that if W is diffeomorphic to a cobordism obtained by "sewing" W_1, W_2, then $Z(W) = Z(W_2) \circ Z(W_1)$



$$\begin{tikzcd} Z(M_1) \arrow[r, "Z(W_1)"] \arrow[dr, "Z(W)"'] & Z(M_2) \arrow[d, "Z(W_2)"] \\ & Z(M_3) \end{tikzcd}$$

We assume that $Z(M)$ or $Z(W)$ depend only on the <u>diffeomorphism type</u> of M or W.

<u>Theorem</u> A 2-dimensional TFT Z determines and is determined by a commutative Frobenius algebra, that is, a commutative unital associative algebra A with a linear function $tr: A \to \mathbb{C}$ such that $\langle x,y\rangle = tr(xy)$ is a nondegenerate inner product.

<u>Sketch of proof</u> of the "determines" part:
(the "is determined by" part is actually somewhat subtle)

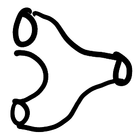
Set $A = Z(S^1)$.

 $\rightsquigarrow$ unit $\mathbb{C} \to A$

$\emptyset \quad S^1$

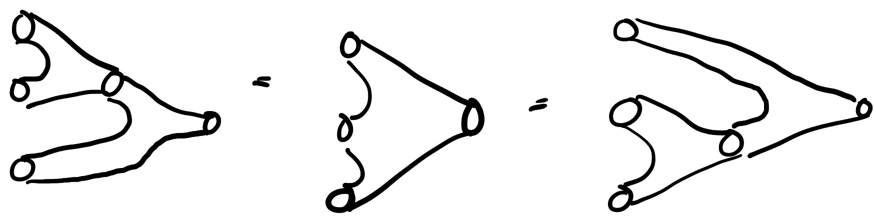
 $tr: A \to \mathbb{C}$

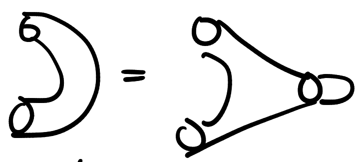
$\emptyset \to S^1$

 product $\bullet : A \otimes A \to A$

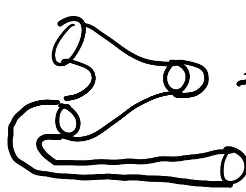
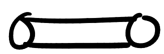
$S^1 \amalg S^1 \to S^1$

Associativity:



 $\langle a,b\rangle = tr(ab) : A \otimes A \to \mathbb{C}$

$S^1 \amalg S^1 \to \emptyset$

 $=$  $= Id$

proves that pairing is nondegenerate.

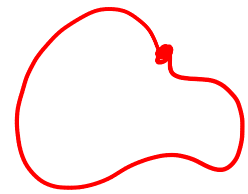
String theory interpretation: Think of these surfaces as <u>world sheets</u> of strings propagating in some target space X.

The vectorspace $\mathcal{H} = Z(S^1)$ is the Hilbert space of <u>closed string states</u>.
The maps $Z(W): \mathcal{H}^{\otimes a} \to \mathcal{H}^{\otimes b}$ represent transition amplitudes for various physical processes.

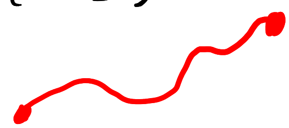
$a \Big\{$  $\Big\} b$

The "fields" in this theory are maps $u: \Sigma \to X$ where Σ is the surface.

So far we have considered closed strings (loops)

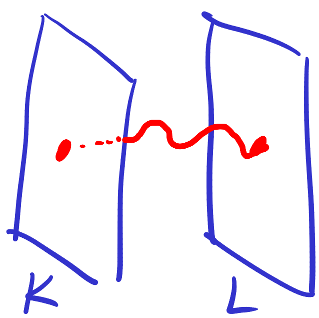


We may also consider open strings (arcs)



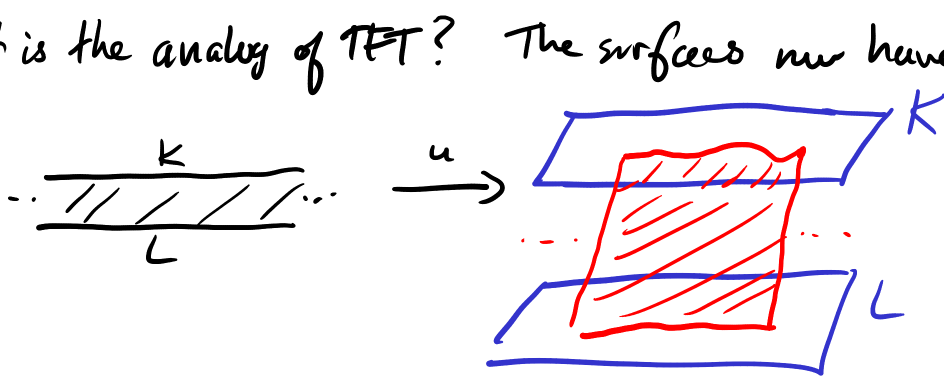
In a reasonable theory, we should prescribe the boundary conditions, the behavior of the open string at the endpoints.

For instance, Dirichlet boundary conditions = conditions on the location of the endpoints. We choose subspaces $K, L \subseteq X$ and require that the open strings end on them.



K and L will need to satisfy certain conditions and possess extra structure in order to get a reasonable theory. They are then called <u>D-branes</u>.

What is the analog of TFT? The surfaces now have boundary.

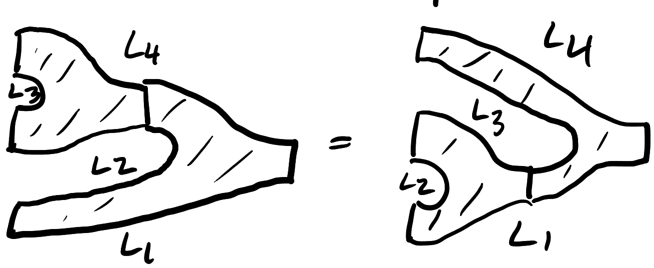


And the boundary components are labeled by D-branes.

We have vector spaces $\mathcal{H}_{KL}$ = space of states of open strings starting on K and ending on L.

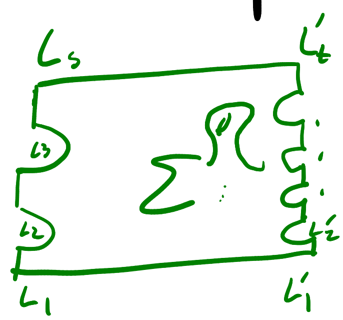
And maps

$: \mathcal{H}_{L_1L_2} \otimes \mathcal{H}_{L_2L_3} \to \mathcal{H}_{L_1L_3}$



Associativity: The boundary conditions form a $\mathbb{C}$-linear category!

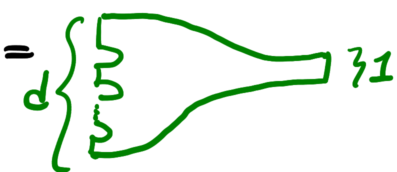
In real examples, $\mathcal{H}_{KL}$ is actually the cohomology of a cochain complex $(\mathcal{C}_{KL}, Q)$, and at the chain level, the maps $Z(\Sigma)$ depend not only on the diffeomorphism type of Σ but also on a complex (or even Riemannian) structure on Σ. We then get chain maps.



$$C_*(\overline{\mathcal{M}}_\Sigma) \otimes \mathcal{C}_{L_1L_2} \otimes \cdots \otimes \mathcal{C}_{L_{s-1}L_s} \longrightarrow \mathcal{C}_{L'_1L'_2} \otimes \cdots \otimes \mathcal{C}_{L'_{t-1}L'_t}$$

Moduli space of complex structures on Σ

Restricting to the cases where

$\Sigma =$ 

We have $\overline{\mathcal{M}}_\Sigma = \overline{\mathcal{R}}^{d+1}$.

The maps are compatible with gluing, which reproduces the A_∞-associativity equations.

Thus we see that the notion of "open string theory with D-branes" includes the structure of an A_∞-category whose objects are the D-branes themselves.