Lecture 10 Moduli spaces and operads.

Last time we discussed $\overline{\mathcal{R}}^{d+1} = \coprod_T \mathcal{R}^T$, the moduli space of stable $(d+1)$-pointed disks, whose strata are labeled by d-leafed trees.
Today we will connect this space to the A_∞-associativity equations.

Since $\overline{\mathcal{R}}^{d+1} = \coprod_T \mathcal{R}^T$ and $\mathcal{R}^T = \prod_{v \in Ve(T)} \mathcal{R}^{\deg(v)}$

we see that the spaces $\{\overline{\mathcal{R}}^{d+1}\}_{d=2}^{\infty}$ have a recursive structure, where the boundary of one space is composed of products of lower dimensional spaces in the same family.
This can be formalized as follows

Operads (J.P. May) Let $\mathcal{C}$ be a (symmetric monoidal) category, e.g. Sets, Top. spaces, Vector spaces, cochain complexes.
An operad in $\mathcal{C}$ is a collection of objects, $\{\mathcal{O}(n)\}_{n=1}^{\infty}$ an element $1 \in \mathcal{O}(1)$, and "composition" maps
$\circ : \mathcal{O}(n) \times \mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_n) \longrightarrow \mathcal{O}(k_1 + \cdots + k_n)$
These must satisfy associativity and identity axioms:
Think of $\mathcal{O}(n)$ as a collection of n-ary operations $f(x_1, \dots, x_n) \in \mathcal{O}(n)$.
$1 \in \mathcal{O}(1)$ is the identity operations.
for $g \in \mathcal{O}(n)$, $f_i \in \mathcal{O}(k_i)$, we can compose

$$\circ(g, f_1, \dots, f_n)(x_{11}, x_{12}, \dots, x_{1k_1}, x_{21}, \dots, x_{2k_2}, \dots, x_{n1}, \dots, x_{nk_n})$$

$$= g(f_1(x_{11}, \dots, x_{1k_1}), f_2(x_{21}, \dots, x_{2k_2}), \dots, f_n(x_{n1}, \dots, x_{nk_n}))$$

Identity axiom: $1 \circ g = g \circ (1, \dots, 1)$

Associativity: $g \circ (f_1 \circ (h_{11}, \dots, h_{1k_1}), \dots, f_n \circ (h_{n1}, \dots, h_{nk_n}))$
$$= (g \circ (f_1, \dots, f_n)) \circ (h_{11}, \dots, h_{1k_1}, \dots, h_{n1}, \dots, h_{nk_n}).$$

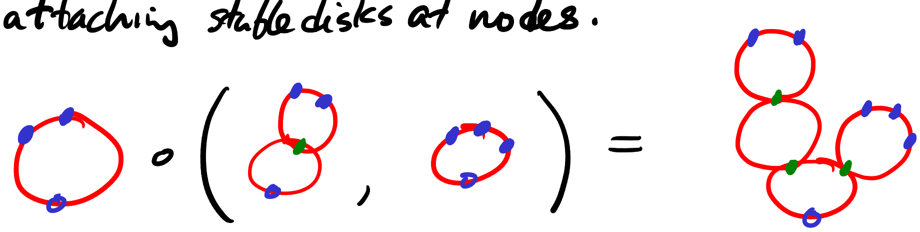
An algebra over an operad is a object A together with maps
$$\mathcal{O}(n) \times \underbrace{A \times \cdots \times A}_{} \to A$$
That are compatible with composition.
Operads let us think about an algebraic theory separately from any given example. An Algebra is a specific model for the axioms that the operad encodes.

Example $\mathcal{C} = \mathbb{C}$-vector spaces. $\mathcal{O}(n) = \mathbb{C}$ for each n.
$\times = \otimes_{\mathbb{C}}$. Then $\mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes \cdots \otimes \mathcal{O}(k_n) \to \mathcal{O}(k_1 + \cdots + k_n)$ is an isomorphism.

An algebra over this operad is an associative $\mathbb{C}$-algebra.
$Id \in \mathcal{O}(1) \cong \mathbb{C}$ is identity.
$m \in \mathcal{O}(2) \cong \mathbb{C}$ is multiplication $(a,b) \mapsto ab$
$t \in \mathcal{O}(3) \cong \mathbb{C}$ is triple product $(a,b,c) \to abc$.
The multiplication is associative because both compositions
$$\underset{m}{\mathcal{O}(2)} \times \underset{m}{\mathcal{O}(2)} \times \underset{1}{\mathcal{O}(1)} \longrightarrow \mathcal{O}(3)$$
$$\underset{m}{\mathcal{O}(2)} \times \underset{1}{\mathcal{O}(1)} \times \underset{m}{\mathcal{O}(2)}$$
$m(m,1)$, $m(1,m)$ are required to be the same $= t$

This is called the <u>associative operad</u> Ass.

Ex The spaces $\{\overline{\mathcal{R}}^{d+1}\}_{d=2}^{\infty}$ form a topological operad
Define $\mathcal{O}(1) = pt = \{Id\}$ $\quad \mathcal{O}(n) = \overline{\mathcal{R}}^{n+1}$ for $n \geq 2$.
Composition is attaching stable disks at nodes.



Now apply the functor $C_*(-)$ (singular chains) to this operad. We get an operad: $\mathcal{O}(n) = C_*(\overline{\mathcal{R}}^{n+1})$ in the category of chain complexes.

Going against our general convention, we will use chain complexes $\partial : C_n \to C_{n-1}$ in this discussion

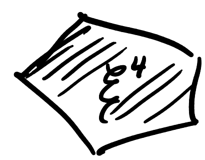
$*$ Now Algebras over $C_*(\overline{\mathcal{R}}^{n+1})$ in chain complexes are essentially A_∞-algebras!

Let us at least see Algebra over $C_*(\overline{\mathcal{R}}^{n+1}) \longrightarrow A_\infty$-algebra.

Let $A = (\oplus A_n, \partial)$ be a chain complex that is an algebra over $C_*(\overline{\mathcal{R}}^{n+1})$, so we have chain maps

$\text{act} : C_*(\overline{\mathcal{R}}^{n+1}) \otimes A^{\otimes n} \longrightarrow A.$

Choose an orientation of $\overline{\mathcal{R}}^{n+1}$, and let ξ_n be the corresponding "fundamental chain" $\xi_n \in C_{n-2}(\overline{\mathcal{R}}^{n+1})$ which has top dimension and degree 1 everywhere.



Now define $\mu^n : A^{\otimes n} \to A$ by

$$\mu^n(a_n, \dots, a_1) = \text{act}(\xi^n, a_n, \dots, a_1)$$

Because ξ^n is <u>not</u> a cycle if $n > 2$, μ^n is <u>not</u> a chain map if $n > 2$.

But $\partial \xi^n =$ sum over codim 1 faces of compositions of lower ξ^k's

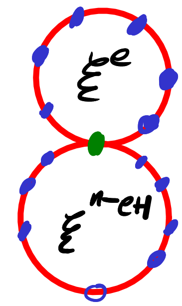
A bit more precisely

$$\partial \xi^n = \sum_{\substack{2 \le e \le n-1 \\ 0 \le i \le d-e}} \pm \; \xi^{n-e+1} \circ (1, \dots, 1, \underset{\substack{\uparrow \\ (i+1)\text{th spot}}}{\xi^e}, 1, \dots, 1).$$

(Note $e \ge 2$, $n-e+1 \ge 2$ in this sum!)

The $\pm$ sign depends on the choices of orientations for $\overline{\mathcal{R}}^{d+1}$.

Why? Codim 1 faces of $\overline{\mathcal{R}}^{d+1} = \coprod_T \mathcal{R}^T$ correspond to when T has one internal edge correspond to when the stable disk has one node.



In terms of μ^n, equation ⊛ becomes

$$(\partial\mu^n)(a_n,\dots,a_1) = \sum_{\substack{2\le e\le n-1\\ 0\le i\le d-e}} \pm\mu^{n-e+1}(a_n,\dots,\mu^e(a_{i+e},\dots,a_{i+1}),a_i,\dots,a_1)$$

$$\|$$

$$\pm\partial(\mu^n(a_n,\dots,a_1)) + \sum \pm\, \mu^n(a_n,\dots,\partial a_i,\dots,a_1)$$

If we set $\mu^1 = \partial$, we can combine both sides into the A_∞-equation.

$$0 = \sum_{1\le e\le n} \pm\,\mu^{n-e+1}(a_n,\dots,\mu^e(a_{i+e},\dots,a_{i+1}),a_i,\dots,a_1)$$

How to determine $\pm$ sign, at least in principle:

Choose orientation for $\overline{\mathcal{R}}^{d+1}$: eg $\mathcal{R}^{d+1}\subset\overline{\mathcal{R}}^{d+1}$ is an open dense subset, so we choose an orientation there.

As $\mathcal{R}^{d+1} = \mathrm{Conf}_{d+1}(\partial\mathbb{D})/\mathrm{Aut}(\mathbb{D})$, can embed $\mathcal{R}^{d+1}\hookrightarrow(\partial\mathbb{D})^{d-2}$ as an open subset. $[z_0,z_1,\dots,z_d]\mapsto[1,i,-1,z_3',\dots,z_d']$

$$\Downarrow$$

$$(z_3',z_4',\dots,z_d')\in(\partial\mathbb{D})^{d-2}$$

Orienting $\partial\mathbb{D}$ as boundary of $\mathbb{D}$, we induce an orientation on the product $(\partial\mathbb{D})^{d-2}$, hence on $\mathcal{R}^{d+1}$.

In equation ⊛ the $\pm$ depends on whether the orientation of $\mathcal{R}^{d-e+1}\times\mathcal{R}^e$ as a product agrees or differs from the orientation induced on it as a boundary of $\mathcal{R}^d$.