Lecture 9 Deligne-Mumford-Stasheff moduli space of stable pointed disks

Recall $\mathbb{D} = \{ |z| \leq 1 \} \subseteq \mathbb{C}$ $\qquad \mathrm{Aut}(\mathbb{D}) \cong PSL(2,\mathbb{R})$

$\partial \mathbb{D} = S^1$.

$\mathrm{Conf}_{d+1}(\partial \mathbb{D})$ configurations of $d+1$ distinct points $(z_0, z_1, \dots, z_d)$ numbered in counterclockwise order.

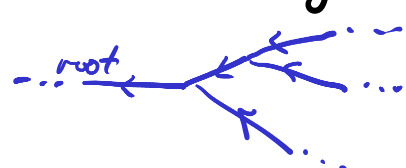
$$\mathcal{R}^{d+1} = \mathrm{Conf}_{d+1}(\partial\mathbb{D}) / \mathrm{Aut}(\mathbb{D}) \qquad \mathcal{S}^{d+1} = \mathrm{Conf}_{d+1}(\partial \mathbb{D}) \times_{\mathrm{Aut}(\mathbb{D})} \mathbb{D}.$$

The projection $\pi : \mathcal{S}^{d+1} \to \mathcal{R}^{d+1}$ admits $d+1$ sections

$$\zeta_k : [z_0, z_1, \dots, z_d] \longmapsto [z_0, z_1, \dots, z_d ; z_k].$$

$\pi : \mathcal{S}^{d+1} \to \mathcal{R}^{d+1}$ is the universal family of $(d+1)$-pointed disks. For compactification, we enlarge the set of objects considered.

<u>Trees</u>: <u>Def</u> A <u>d-leafed tree</u> will mean a properly embedded planar tree $T \subset \mathbb{R}^2$ with $d+1$ semiinfinite edges, one of which is distinguished and called the <u>root</u>, the others are called the <u>leaves</u>.



The semi-infinite edges are called <u>exterior</u>, others are <u>interior</u>.

$Ve(T)$ = set of vertices $\qquad Ed(T)$ = set of edges

$$Ed(T) = Ed^{ext}(T) \amalg Ed^{int}(T).$$

Orient the edges <u>towards the root</u>.

A flag is a pair (v,e) of a vertex and an incident edge.

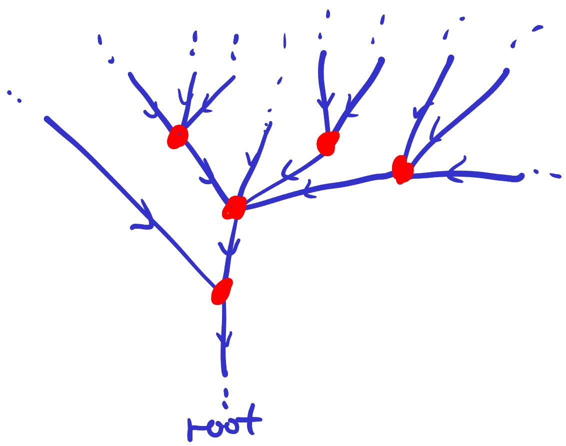
We number the flags at v $\quad 0, 1, \dots, \deg(v) - 1$, where 0 is the outgoing edge, and the numbering is counterclockwise.

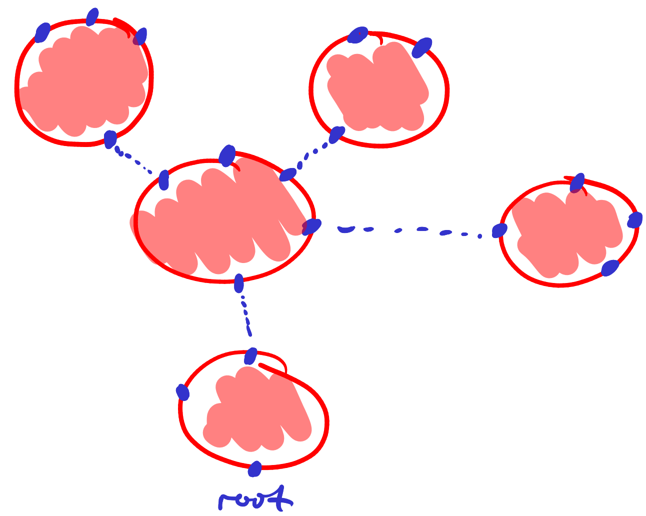
A tree is called <u>stable</u> if $\deg(v) \geq 3$ for all $v \in Ve(T)$

From a stable d-leafed tree T, we can build a class of singular configurations of disks.

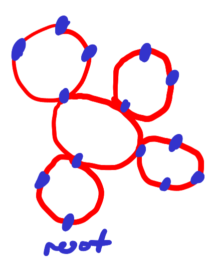
Let T = d-leafed tree. For each $v \in Ve(T)$, choose a $\deg(v)$-pointed disk $S_v = [z_0, z_1, \dots, z_{\deg(v)}]$. Flags at $v \overset{1:1}{\longleftrightarrow}$ marked points.

Each internal edge $e \in Ed^{int}(T)$ starts at some vertex v_0 and ends at v_1. We identify the 0-th marked point of S_{v_0} with the jth marked point of S_{v_1}. As in





Also depicted as



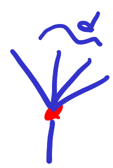
This is called a <u>stable pointed disk</u> of combinatorial type T.

<u>Nodes</u> $\leftrightarrow$ internal edges.

Let $\mathcal{R}^T$ be the moduli space of such. It is clear that

$$\mathcal{R}^T \cong \prod_{v \in Ve(T)} \mathcal{R}^{\deg(v)}.$$

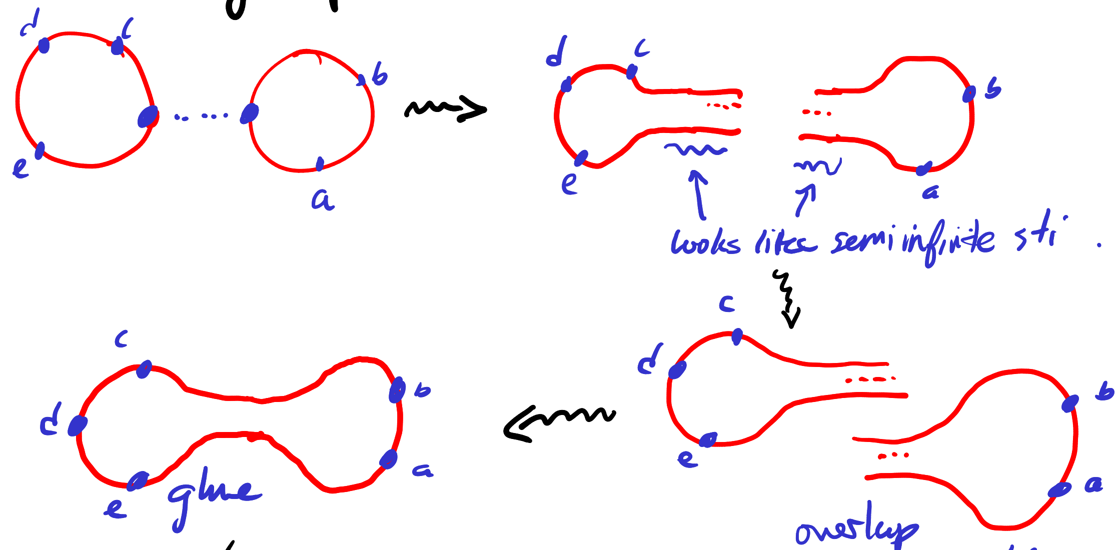
Note that if T =



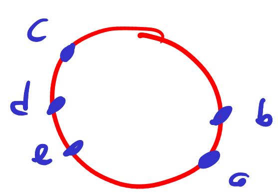
Then $\mathcal{R}^T = \mathcal{R}^{d+1}$

We can patch the $\mathcal{R}^T$ for all d-leafed trees T into a compactification of $\mathcal{R}^{d+1}$!

Gluing: We are going to be a bit imprecise because the main idea is very simple. We want to smooth the nodes.



apply uniformization



overlap by some variable amount

there is a parameter here called the gluing parameter ρ.

$\rho \to 0$ corresponds to infinite length.

Given a stable pointed disk S of combinatorial type T, we can apply the gluing process with parameters $(\rho_e \,;\, e \in Ed^{int}(T))$ to obtain a pointed disk. Suitably defining the gluing parameters ρ_e, we get an embedding

$$\mathcal{R}^T \times (0,1)^{Ed^{int}(T)} \hookrightarrow \mathcal{R}^{d+1} \qquad (T \text{ a } d\text{-leafed tree})$$

We can then attach $\mathcal{R}^T$ to $\mathcal{R}^{d+1}$ by forming the pushout

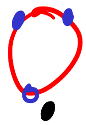
$$\mathcal{R}^T \times [0,1)^{Ed^{int}(T)} \coprod_{\mathcal{R}^T \times (0,1)^{Ed^{int}(T)}} \mathcal{R}^{d+1}.$$

With a bit more care, we can do this for all d-leafed trees simultaneously, and get

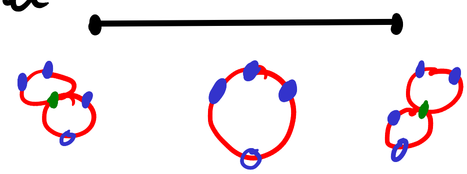
$$\overline{\mathcal{R}}^{d+1} = \coprod_T \mathcal{R}^T \supset \mathcal{R}^{d+1}.$$

This is the Deligne-Mumford-Stasheff compactification.

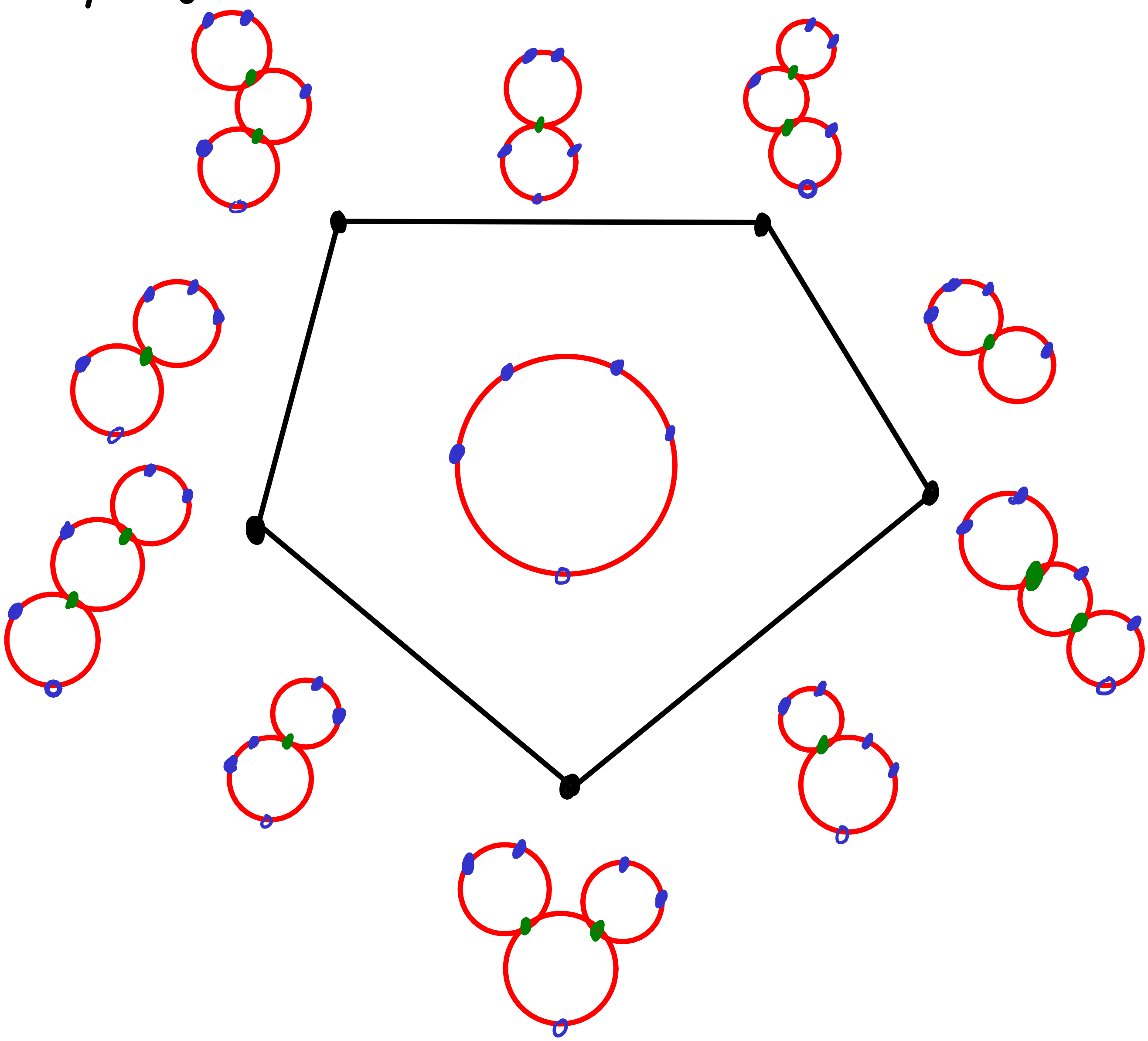
$\overline{\mathcal{R}}^3 = pt$



$\overline{\mathcal{R}}^4$ = interval



$\overline{R}^5$ = pentagon



$\overline{\mathcal{R}}^d$ = a polyhedron also known as the Stasheff associahedron.