Lecture 8 Associativity and Riemann surfaces

Today we take a break from the abstract category theory and talk about some geometry. It is a geometry that "encodes" the A_∞ associativity equations.
→ Will connect to holomorphic curve theory and TFTs.

Associativity $a(bc)=(ab)c$

"Strong associativity": any way of putting parentheses into $a_1 a_2 \cdots a_n$ yields the same result.
e.g. $(a_1((a_2 a_3)a_4))a_5 = a_1(a_2(a_3(a_4 a_5)))$

Of course associativity implies strong associativity.
How many parenthesizations are there?
Let C_n be the number of parenthesizations of a word of length $n+1$

Theorem $C_n = \frac{1}{n+1}\binom{2n}{n}$, the Catalan numbers.

Proof $C_0 = 1$, $C_1 = 1$ clear. For $n \geq 2$, look at the look at the multiplication which is done last. It divides the word of length $n+1$ into words of length $i+1$ and $n-i$, which have C_i and C_{n-i-1} parenthesizations respectively. So

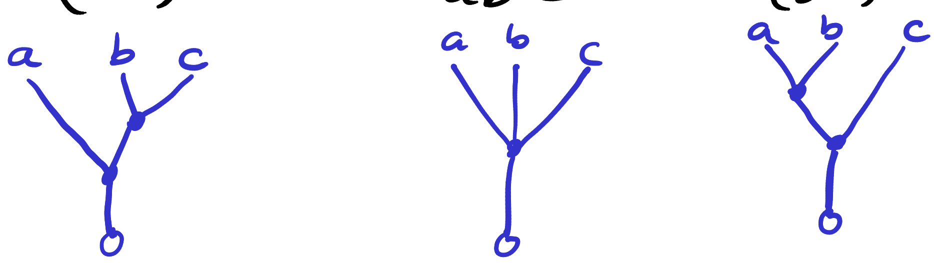
$$C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}$$

It is known that this recursion relation produces the catalan numbers. ∎

The proof suggests it might be interesting to look at partial parenthesizations like $a_1(a_2 a_3 a_4)a_5$

<u>Observation</u>: <u>partial parenthesizations</u> correspond to certain <u>planar trees</u>:

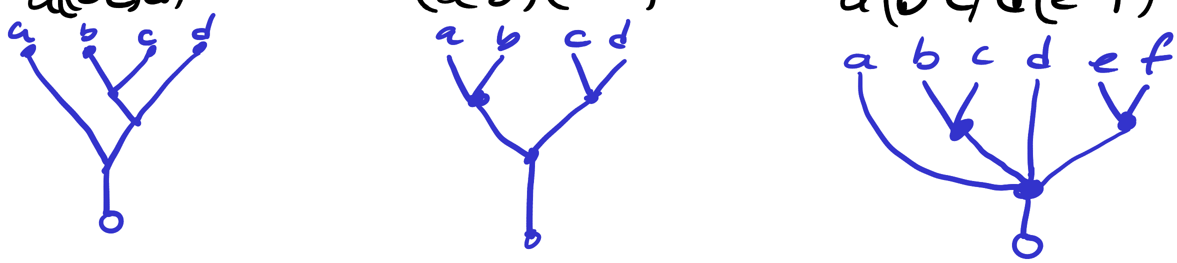
a(bc)   abc   (ab)c



← "input"

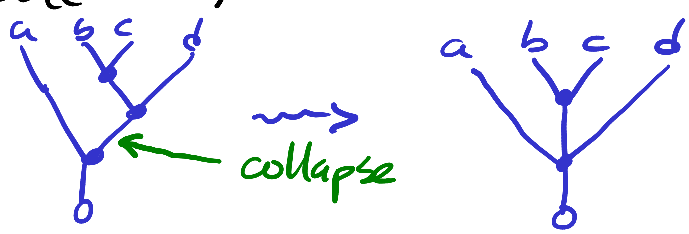
← "root vertex"
"output"

a((bc)d)   (ab)(cd)   a(bc)d(ef)



Observe that "removing a set of parentheses" collapses an edge

a((bc)d) ⇝ a(bc)d



So pairs of parentheses correspond to internal edges (edges not touching a leaf.)

Now, the same combinatorics appears when studying <u>degenerations of Riemann surfaces.</u>

Let $\mathbb{D} = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$ be the complex unit disk.

$\partial \mathbb{D} = S^1$, the unit circle.

The holomorphic automorphism group of $\mathbb{D}$ is isomorphic to $PSL(2,\mathbb{R})$. More precisely, $\mathbb{D} \cong \mathbb{H} = \{ z \mid \operatorname{Im} z \geq 0 \}$

$$\operatorname{Aut}(\hat{\mathbb{C}}) = PSL(2,\mathbb{C}) = \left\{ \varphi \;\middle|\; \varphi(z) = \frac{az+b}{cz+d} \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}) \right\}$$

$$\cup$$

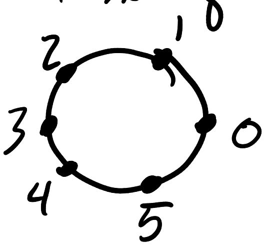
$$\operatorname{Aut}(\mathbb{H}) = PSL(2,\mathbb{R})$$

An explicit isomorphism $\psi : \mathbb{D} \to \mathbb{H}$ is $\psi(z) = \dfrac{1 - iz}{z - i}$

So $\quad Aut(\mathbb{D}) = \psi^{-1} Aut(\mathbb{H}) \psi$

is a conjugate of $PSL(2, \mathbb{R})$ inside $PSL(2, \mathbb{C})$.

Let $Conf_{n+1}(\mathbb{D}, \partial\mathbb{D})$ be the set of configurations of $n+1$ marked points on $\partial\mathbb{D} = S^1$.



We label the points $\{z_0, z_1, \ldots, z_d\}$ in counter-clockwise order.

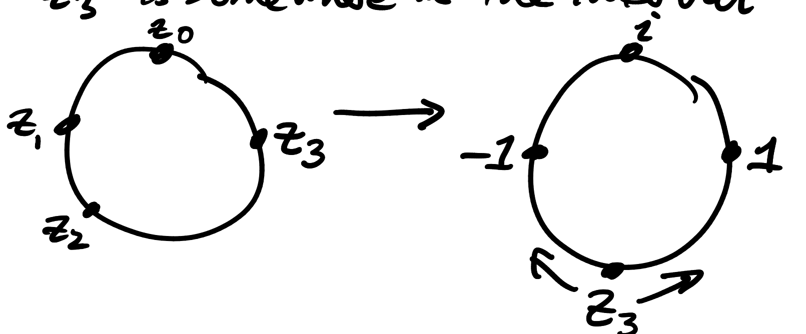
The points are required to be <u>distinct</u>.

Set $\quad R_{n+1} = Conf_{n+1}(\mathbb{D}, \partial\mathbb{D}) / Aut(\mathbb{D})$.

This is the <u>moduli space of disks with $d+1$ boundary marked points</u>

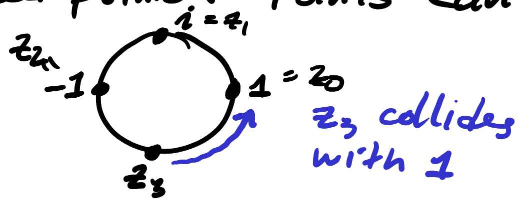
<u>Fact</u>: $R_3 = pt$. More precisely, any triple z_1, z_2, z_3 of pairwise distinct, counterclockwise-ordered points in $\partial\mathbb{D}$ can be brought to $1, i, -1$ by some $\varphi \in Aut(\mathbb{D})$

R_4 is homeomorphic to an interval. Proof: bring z_0, z_1, z_2 to $1, i, -1$. Then z_3 is somewhere in the interval between -1 and 1.

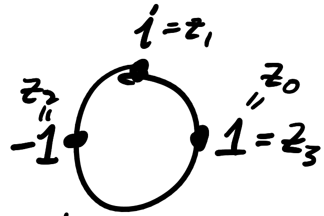


More generally, R_{d+1} is homeomorphic to $\mathbb{R}^{d-2}$.

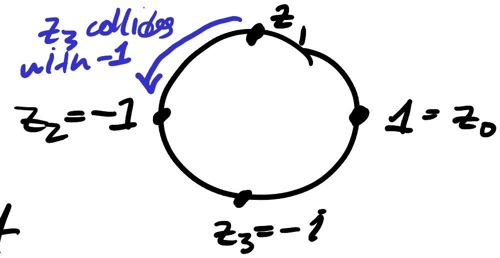
<u>Compactification</u>: What are the possible limiting behaviors of families of disks with marked points? Points can "collide". For instance in R_4



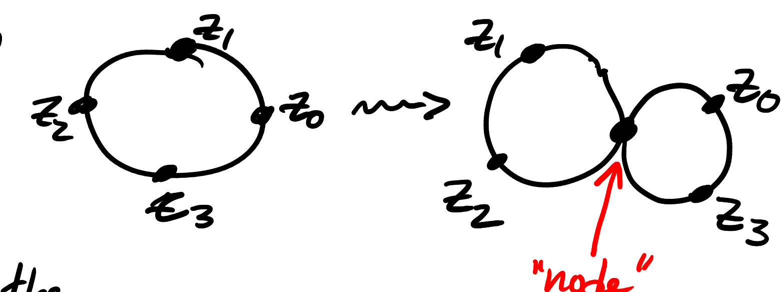
One might decide to have the limiting configuration be where $z_0 = z_3$. But this is biased toward the choice of coordinates where z_0, z_1, and z_2 are held fixed at $1, i, -1$.



There is another coordinate system where, say z_0, z_3, z_2 are held fixed, and z_1 moves. In this coordinate system, The picture looks like



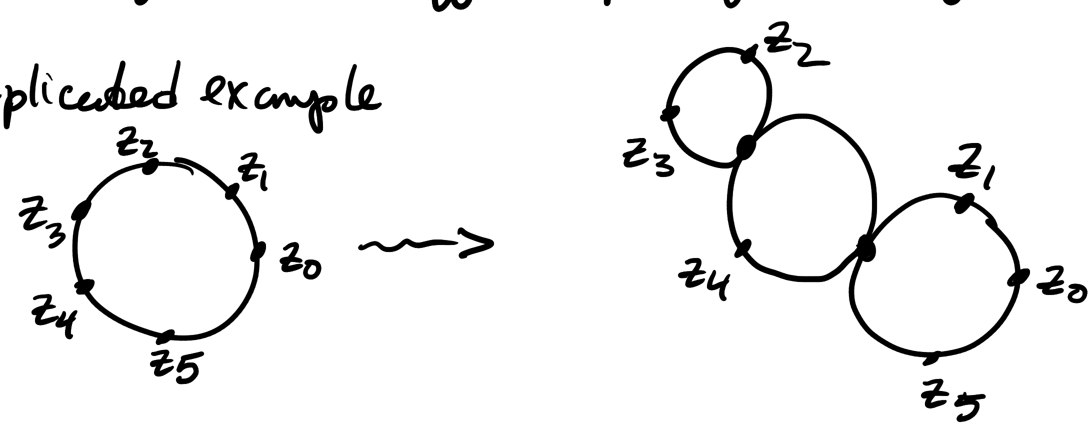
But now it appears that the limit should be $z_1 = z_2$, and z_0 and z_3 remain separated. A way to give an unbiased answer is to allow the disk to split in two



This is the basic idea of the

<u>Deligne-Mumford-Stasheff compactification</u> of R_{d+1}.

More complicated example



The combinatorics of these degenerations reproduce the trees we saw before in the analysis of associativity!