Lecture 7:

A_∞-categories:

We now build the theory of A_∞-categories. There is a subtlety with unit morphisms that we will come to, but we begin by ignoring them.

Let k be a field

Def A <u>non-unital A_∞-category</u> $\mathcal{A}$ consists of a set $Ob\,\mathcal{A}$ of objects, for each pair of objects X_0, X_1, a graded k-vectorspace $hom_{\mathcal{A}}(X_0, 1)$ and composition maps $\{\mu^d_{\mathcal{A}}\}_{d\geq 1}$

$$\mu^d_{\mathcal{A}} : hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes hom_{\mathcal{A}}(X_0, X_1) \to hom_{\mathcal{A}}(X_0, X_d)[2-d]$$

These are required to satisfy the A_∞-associativity equations

$$\sum_{e,i} (-1)^* \mu^{d-e+1}_{\mathcal{A}}(a_d, \ldots, a_{i+e+1}, \mu^e(a_{i+e}, \ldots, a_{i+1}), a_i, \ldots, a_1) = 0$$

where $* = |a_1| + \cdots + |a_i| - i$, and the sum is over all terms of this shape: $1 \leq e \leq d$, $0 \leq i \leq d-e$.

$d=1$ $\quad \mu^1_{\mathcal{A}}(\mu^1_{\mathcal{A}}(a_1)) = 0$ $\quad$ so $(hom_{\mathcal{A}}(X_0, X_1), \mu^1)$ is a complex

$d=2$ $\quad \mu^1(\mu^2(a_2, a_1)) + \mu^2(a_2, \mu^1(a_1)) + (-1)^{|a_1|-1}\mu^2(\mu^1(a_2), a_1) = 0$

<u>Remark:</u> DG-categories correspond to A_∞-categories with $\mu^d_{\mathcal{A}} = 0$ for $d \geq 3$. μ^1 corresponds to the differential, and $\mu^2_{\mathcal{A}}$ to the product (with slightly altered signs).

$$d(a) = (-1)^{|a|}\mu^1(a) \qquad a_2 \cdot a_1 = (-1)^{|a_1|}\mu^2(a_2, a_1).$$

The cohomology category $H^*(\mathcal{A})$ has the same objects as $\mathcal{A}$ and morphisms $H^*(\hom_{\mathcal{A}}(X_0, X_1), \mu^1_{\mathcal{A}})$. The composition is

$$[a_2]\cdot[a_1] = (-1)^{|a_1|}[\mu^2_{\mathcal{A}}(a_2, a_1)]$$

__Definition__ We say $\mathcal{A}$ is __c-unital__ (cohomologically unital) if $H^*(\mathcal{A})$ has identity morphisms (which it ought to in order to be a category in the usual sense.)

__Remark__: Other notions of unitality exist: strict unitality, and homotopy unitality. It turns out they are not substantially different.

__Definition__ A non-unital A_∞-functor $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ between non-unital A_∞-categories $\mathcal{A}$ and $\mathcal{B}$ consists of a map $\mathcal{F}^0: \mathrm{Ob}\,\mathcal{A} \to \mathrm{Ob}\,\mathcal{B}$ and for each $d \geq 1$ a map

$$\mathcal{F}^d: \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1) \to \hom_{\mathcal{B}}(\mathcal{F}^0(X_0), \mathcal{F}^0(X_1))[1-d]$$

such that:

$$\sum_r \sum_{s_1, \ldots, s_r} \mu^r_{\mathcal{B}}(\mathcal{F}^{s_r}(a_d, \ldots, a_{d-s_r+1}), \ldots, \mathcal{F}^{s_1}(a_{s_1}, \ldots, a_1))$$

$$= \sum_{m,n} (-1)^* \mathcal{F}^{d-m+1}(a_d, \ldots, a_{n+m+1}, \mu^m_{\mathcal{A}}(a_{n+m}, \ldots, a_{n+1}), a_n, \ldots, a_1)$$

where as before $* = |a_1| + \cdots + |a_n| - n$ and $r \geq 1$, $s_1 + s_2 + \cdots + s_r = d$

There is an induced functor $H^*(\mathcal{F}) : H^*(\mathcal{A}) \to H^*(\mathcal{B})$
Such that $[a] \mapsto [\mathcal{F}^1(a)]$ on morphisms.

We call $\mathcal{F}$ a <u>quasi-isomorphism</u> if $H^*(\mathcal{F})$ is an isomorphism of (not necessarily unital) categories.

If $\mathcal{A}$ and $\mathcal{B}$ are c-unital, we call $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ c-unital if $H^*(\mathcal{F}) : H^*(\mathcal{A}) \to H^*(\mathcal{B})$ is unital (so that it is an "honest" functor between "honest" categories.)

When $\mathcal{A}$, $\mathcal{B}$, $\mathcal{F}$ are c-unital, we call $\mathcal{F}$ a <u>quasi-equivalence</u> if $H^*(\mathcal{F}) : H^*(\mathcal{A}) \to H^*(\mathcal{B})$ is an equivalence of categories.

<u>Remark</u>: In practice, we will formulate HMS as a statement saying two c-unital A_∞-categories are quasi-equivalent.

<u>More about A_∞-functors:</u> A_∞-functors can be composed:

$$\mathcal{F} : \mathcal{A} \to \mathcal{B} \qquad \mathcal{G} : \mathcal{B} \to \mathcal{C} \qquad \mathcal{G} \circ \mathcal{F} : \mathcal{A} \to \mathcal{C}$$

$$(\mathcal{G} \circ \mathcal{F})^d (a_d, \dots, a_1) = \sum_{r \geq 1} \sum_{s_1 + \dots + s_r = d} \mathcal{G}^r(\mathcal{F}^{s_r}(a_d, \dots, a_{d-s_r+1}), \dots, \mathcal{F}^{s_1}(a_{s_1}, \dots, a_1))$$

This composition is strictly associative, and the Identity functor is an Identity element for it.

Interpretation of A_∞-functor equations: $\mathcal{F}^0 : \mathrm{Ob}\,\mathcal{A} \to \mathrm{Ob}\,\mathcal{B}$

$$\mathcal{F}^1 : \hom_{\mathcal{A}}(X_0, X_1) \to \hom_{\mathcal{B}}(\mathcal{F}^0 X_0, \mathcal{F}^0 X_1)$$

Ordinarily, $\mathcal{F}^1$ would preserve composition.

$$\mathcal{F}^1(a_2 \cdot a_1) = \mathcal{F}^1(a_2) \cdot \mathcal{F}^1(a_1)$$

But now $\mathcal{F}^1$ only preserves composition up to homotopy

$\underline{d=1}$: $\quad \mu^1_{\mathcal{B}}(\mathcal{F}^1(a_1)) = \mathcal{F}^1(\mu^1_{\mathcal{A}}(a_1))$ so $\mathcal{F}^1$ is chain map

$\underline{d=2}$: $\quad \mu^1_{\mathcal{B}}(\mathcal{F}^2(a_2,a_1)) + \mu^2_{\mathcal{B}}(\mathcal{F}^1(a_2),\mathcal{F}^1(a_1))$

$$= \mathcal{F}^1(\mu^2_{\mathcal{A}}(a_2,a_1)) + \mathcal{F}^2(a_2,\mu^1_{\mathcal{A}}(a_1)) + (-1)^{|a_1|-1}\mathcal{F}^2(\mu^1_{\mathcal{A}}(a_2),a_1)$$

$\underline{\text{Observe}}$: if $\mathcal{A}$ and $\mathcal{B}$ are DG algebras (A_∞-cats with one object and $\mu^d = 0$ for $d \geq 3$)
Then there are more A_∞-morphisms $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ than there are DG-algebra morphisms (DG algebra morphism = Algebra morphism which is also a chain map = $\mathcal{F}$ such that $\mathcal{F}^d = 0$ for $d \geq 2$.)

$\underline{\text{Natural Transformations}}$: Let $\mathcal{F}_0, \mathcal{F}_1 : \mathcal{A} \to \mathcal{B}$ be two (non-unital) A_∞-functors. A degree g pre-natural-transformation $T: \mathcal{F}_0 \to \mathcal{F}_1$ is a collection $(T^0, T^1, T^2, \ldots)$

$$T^d : \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1) \to \hom_{\mathcal{B}}(\mathcal{F}_0^0 X_0, \mathcal{F}_1^0 X_d)[g-d]$$

Denote the collection of such by $\hom^g(\mathcal{F}_0, \mathcal{F}_1)$
There is a differential $\mu^1 : \hom^g(\mathcal{F}_0,\mathcal{F}_1) \to \hom^{g+1}(\mathcal{F}_0,\mathcal{F}_1)$ and higher compositions $\mu^2, \mu^3, \ldots$ that make functors into an A_∞-category nu-fun$(\mathcal{A},\mathcal{B})$. Unital functors are a full subcategory. (See Seidel's book for formulas)
Closed elements $\mu^1(T) = 0$ are the natural transformations, and the coboundaries give a notion of homotopy between them.

$\underline{\text{Theorem}}$: A quasi-equivalence has an inverse up to homotopy.