Lecture 6: A_∞ - algebras and categories

We introduced DG algebras and categories last time, but for HMS we need a bit more generality. A_∞-algebras and categories are a "mild" generalization in the sense that

- Every DG-category is an A_∞-category
- Every A_∞-category is equivalent to some DG category (where "equivalence" is taken in the sense appropriate to A_∞-categories.)

However, the generalization is <u>not mild at all</u> from the point of view of actually writing down the definition, as an A_∞-category has an infinite hierarchy of so-called higher compositions; it is not even actually a category in the MacLane sense!

So why do this? There are several reasons, both technical and conceptual.

1. In a DG category, composition is associative. But the hom-spaces are chain complexes, which are "homotopical" objects so it makes sense to weaken the associativity axiom to "homotopy associativity".
2. In HMS, our ultimate goal is to prove two categories are equivalent. What this will involve is understanding the deformation theory and homotopy theory of the two categories well enough that we can compare them. We shall see that the homotopy theory of A_∞-categories works a bit better than that of DG-categories.

3. One of the categories, the Fukaya category, comes from symplectic geometry, and what the geometry gives you is an A_∞-structure, not a DG-structure. So there is a geometric motivation.
4. The preceding connects to the physical origin of Mirror symmetry, which is a duality between two topologically twisted string theories. The categories we are considering are the "boundary conditions" (D-branes) in these theories, the hom complexes are the "BRST complexes of open string states", and the composition comes from the structure of a 2D Topological Field Theory. It turns out that "Boundary conditions in a 2D Topological Field Theory" naturally forms an A_∞-category.

From DG to A_∞: Homotopy associativity.
We focus on the "algebras" case ("categories" with 1 object)

Let k be a comm. ring, $A = \bigoplus_{n\in\mathbb{Z}} A^n$ a graded k-module.
Let A have a differential $d: A \to A$, $d(A^n) \subseteq A^{n+1}$, $d^2 = 0$.
Suppose $m: A\otimes A \to A$ is a bilinear operation, not assumed associative, which is homogeneous: $m(A^n \otimes A^m) \subseteq A^{n+m}$.
Now, $A\otimes A$ is naturally a complex $(A\otimes A)^n = \bigoplus_{p+q=n} A^p \otimes A^q$

$$d(a\otimes b) := d(a)\otimes b + (-1)^{|a|} a\otimes d(b)$$

So it makes sense to require that $m: A\otimes A \to A$ be a chain map, meaning

$$d(m(a\otimes b)) = m(d(a\otimes b)) = m(d(a)\otimes b) + (-1)^{|a|} m(a\otimes d(b))$$

this is the graded Leibniz rule. It gaurantees that

m descends to a well-defined bilinear operation
$\overline{m} : H^*(A) \otimes H^*(A) \to H^*(A)$.

If we next impose the condition that m be associative:

$$m(a \otimes m(b \otimes c)) = m(m(a \otimes b) \otimes c)$$

we get that (A, d, m) is a DG-algebra.

But instead, we could impose the associativity <u>up to</u> homotopy. We have a natural chain map

$$\text{Associator}_3 : \begin{array}{l} A \otimes A \otimes A \longrightarrow A \\ A^p \otimes A^q \otimes A^r \longrightarrow A^{p+q+r} \\ (a, b, c) \mapsto m(a, m(b,c)) - m(m(a,b), c) \end{array}$$

And we can ask that Associator_3 be chain homotopic to 0.
That is, we can ask that there be a map

$$P : \begin{array}{l} A \otimes A \otimes A \longrightarrow A \\ A^p \otimes A^q \otimes A^r \longrightarrow A^{p+q+r-1} \end{array}$$

Such that

$\text{Associator}_3 = dP + Pd$, that is

$$\begin{aligned} \text{Associator}_3(a,b,c) = dP(a,b,c) + & P(da, b, c) \\ & + (-1)^{|a|} P(a, db, c) \\ & + (-1)^{|a|+|b|} P(a, b, dc) \end{aligned}$$

This will gaurantee that $\overline{m} : H^*(A) \otimes H^*(A) \to H^*(A)$ is associative even if m itself is not.
(Associator_3 (closed elements) = exact element)

When working with such objects, it is better to include P as part of the data (rather than merely postulating its existence).

Then there is another associativity question: it is possible to construct a chain map

$$\text{Associator}_4 : A \otimes A \otimes A \otimes A \to A$$
$$A^p \otimes A^q \otimes A^r \otimes A^s \to A^{p+q+r+s-1}$$

By combining the five terms at right with appropriate signs

$$\begin{cases} \pm P(m(w,x),y,z) \\ \pm P(w, m(x,y), z) \\ \pm P(w,x,m(y,z)) \\ \pm m(P(w,x,y),z) \\ \pm m(w, P(x,y,z)) \end{cases}$$

We would also like this "higher associator" to be null homotopic, meaning we pick $Q : A^{\otimes 4} \to A$ such that

$$\text{Associator}_4 = dQ - Qd.$$

We make Q part of the data, then there is Associator_5, and so on. The end result of this process is the notion of an A_∞-algebra.

We now give the definition of this object (using Seidel's sign conventions)

<u>Def</u> An <u>A_∞-algebra</u> over k consists of a graded k-module $A = \bigoplus_{n \in \mathbb{Z}} A^n$, and a family of multilinear maps $\{\mu^d\}_{d=1}^{\infty}$

$$\mu^d : A^{\otimes d} \to A[2-d] \quad \left(\mu^d(A^{n_1} \otimes A^{n_2} \otimes \cdots \otimes A^{n_d}) \subseteq A^{\sum_{i=1}^d n_i + 2 - d}\right)$$

such that the <u>A_∞-equations</u> hold: for each d, $a_1, a_2, \cdots a_d \in A$,

$$\sum_{e,i} (-1)^* \mu^{d-e+1}(a_d, \dots, a_{i+e+1}, \mu^e(a_{i+e}, \dots, a_{i+1}), a_i, \dots, a_1) = 0$$

where $* = |a_1| + \cdots + |a_i| - i$.