Lecture 5: DG categories

So far we have considered

modules	mod-R	Abelian category
complexes	$Ch(R)$	???
homotopy cat.	$K(R)$	Triangulated category
Derived cat.	$D(R)$	(Localization of) Triangulated category.

What kind of category is $Ch(R)$? More broadly, what kind of category do chain complexes form?
What are the morphism?

$$\mathrm{Hom}_{Ch(R)}(M,N) = \{ f^i : M^i \to N^i \mid d_N^{i+1} f^i = f^{i+1} d_M^i \}$$

But we could also admit all maps (not just chain maps) and allow them to shift degree

$$\mathrm{hom}^p(M,N) := \{ (f^i : M^i \to N^{i+p})_{i\in\mathbb{Z}} \mid \text{no condition} \}$$

The collection of k-modules $\{\mathrm{hom}^p(M,N)\}_{p\in\mathbb{Z}}$, is itself a complex! The differential δ is

$$\delta : \mathrm{hom}^p(M,N) \to \mathrm{hom}^{p+1}(M,N)$$

$$(\delta f)^i = d_N^{i+p} f^i - (-1)^p f^{i+1} d_M^i$$

or, $\delta f = df - (-1)^{|f|} fd$, more compactly.

There is also quite clearly a composition

$$\mathrm{hom}^q(N,Q) \times \mathrm{hom}^p(M,N) \to \mathrm{hom}^{p+q}(M,Q)$$

these structures form what is called a Differential Graded Category.

Analogy. • A monoid is the "same" as a category with one object. So a category is a "monoid with many objects"

• A k-algebra is the "same" as a k-linear category with one object, so a k-linear category is a "k-algebra with many objects."

Indeed if $\mathcal{C}$ is a k-linear cat, then $\mathrm{Hom}_{\mathcal{C}}(X,X)$ is a k-algebra, and in fact, if $S \subseteq \mathrm{Ob}\,\mathcal{C}$ is any set of objects,

$$R(S) = \bigoplus_{X,Y \in S} \mathrm{Hom}_{\mathcal{C}}(X,Y) \text{ is a } k\text{-algebra.}$$

(define composition to be 0 if morphisms aren't composable)

· So a differential graded category must be a "differential graded algebra with many objects"!

Def A differential graded k-algebra is a k-algebra A, with a decomposition $A = \bigoplus_{n\in\mathbb{Z}} A^n$ such that $A^n \cdot A^m \subseteq A^{n+m}$

and a k-linear function $d : A \to A$ such that $d(A^n) \subseteq A^{n+1}$, $d^2 = 0$, and such that the graded Leibniz rule holds:

$$d(ab) = d(a)b + (-1)^n a\,d(b) \quad \text{for } a \in A^n, b \in A^m.$$

Example $\Omega^{\bullet}(M)$, differential forms on a manifold

Def A differential graded k-linear category (DG category) $\mathcal{C}$ consist of a collection of objects, and for each pair of objects X, Y a k-module $\mathrm{hom}_{\mathcal{C}}(X,Y) = \bigoplus_{n\in\mathbb{Z}} \mathrm{hom}^n_{\mathcal{C}}(X,Y)$,

a composition $\circ : \hom^q_{\mathcal{C}}(Y,Z) \times \hom^p_{\mathcal{C}}(X,Y) \to \hom^{p+q}(X,Y)$
and a k-linear differential $d^p_{X,Y} : \hom^p(X,Y) \to \hom^{p+1}(X,Y)$
such that $d^{p+1}_{X,Y} \circ d^p_{X,Y} = 0$ $\quad (d^2 = 0)$
and for any $f \in \hom^q(Y,Z)$, $g \in \hom^p(X,Y)$ we have

$$d^{p+q}_{X,Z}(f \circ g) = d^q_{Y,Z}(f) \circ g + (-1)^q f \circ d^p_{X,Y}(g)$$

i.e. $d(f\circ g) = d(f)\circ g + (-1)^{|f|} f \circ d(g)$

<u>Example</u> let $\mathcal{C}h(R)$ (note script style) be the category with objects complexes of R-modules, and let $\hom^p(M,N)$ and $d = \delta : \hom^p(M,N) \to \hom^{p+1}(M,N)$ be as above. then $\mathcal{C}h(R)$ is a differential graded category

Any DG category $\mathcal{C}$ has associated several ordinary categories

$Z^0(\mathcal{C})$: same objects, $\mathrm{Hom}_{Z^0(\mathcal{C})}(X,Y) = \{ f \in \hom^0(X,Y) \mid df = 0 \}$
$= Z^0(\hom^*(X,Y))$
$=$ closed, degree zero morphisms

$H^0(\mathcal{C})$: same objects, $\mathrm{Hom}_{H^0(\mathcal{C})}(X,Y) = H^0(\hom^*(X,Y))$
$=$ zeroth cohomology of hom complexes

$H^*(\mathcal{C})$: same objects, $\mathrm{Hom}_{H^*(\mathcal{C})}(X,Y) = \bigoplus_{p \in \mathbb{Z}} H^p(\hom^*(X,Y))$
total cohomology.
(this is a <u>graded category</u>)

The category $H^0(\mathcal{C})$ is called the <u>homotopy category of $\mathcal{C}$</u>.

<u>Ex</u> $\mathcal{Ch}(R)$ = DG category of complexes of R-modules.
$Z^0(\mathcal{Ch}(R)) = Ch(R)$ "ordinary" category of complexes
$H^0(\mathcal{Ch}(R)) = K(R)$ homotopy category of complexes.

<u>Rmk</u>: What about $D(R)$? In fact, it is possible to obtain $D(R)$ as the homotopy category of a DG category

<u>Before</u>: $$Ch(R) \xrightarrow[\text{category}]{\text{homotopy}} K(R) \xrightarrow{\text{localize}} D(R)$$

<u>DG world</u>: $$\mathcal{Ch}(R) \xrightarrow[\text{quotient}]{\text{Keller-Drinfeld}} \frac{\mathcal{Ch}(R)}{\text{Acyclic}} \xrightarrow[\text{category}]{\text{homotopy}} D(R)$$

In the DG world, we can interchange the order of the steps, and do the localization first. This is called the Keller or Drinfeld quotient of a DG-category.

<u>Q</u>: What is a "triangulated" DG-category? As we will later see, to be "triangulated" is a <u>property</u> of a DG-category, it does not involve specifying as data the collection of distinguished triangles. So a DG-category is either "triangulated" or it isn't. $\mathcal{Ch}(R)$ is triangulated in this sense. This is one reason why DG categories are preferred to "plain" triangulated categories in many cases.