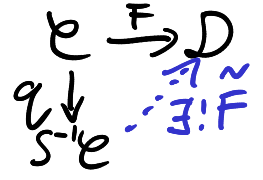


Lecture 4: [Finish previous notes, then:]

The derived category $D(R)$ is formed from the homotopy category $K(R)$ of cochain complexes of R -modules by "inverting quasi-isomorphisms". This is a process analogous to the localization of a ring (for instance, the field of fractions of an integral domain.)

Def let S be a collection of morphisms in a category \mathcal{C} . A localization of \mathcal{C} w.r.t. S is a category $S^{-1}\mathcal{C}$ and a functor $q: \mathcal{C} \rightarrow S^{-1}\mathcal{C}$ such that

- $\forall s \in S, q(s)$ is an isomorphism in $S^{-1}\mathcal{C}$
- If $F: \mathcal{C} \rightarrow \mathcal{D}$ is a functor such that $F(s)$ is an isomorphism in \mathcal{D} for all $s \in S$, then $\exists!$ $\tilde{F}: S^{-1}\mathcal{C} \rightarrow \mathcal{D}$ such that $F = \tilde{F} \circ q$



Because this is a universal property, $S^{-1}\mathcal{C}$ is unique if it exists.

Def let $\mathcal{Q} = \{ f: X \rightarrow X' \mid f \text{ is quasi-iso} \} \subset K(R)$
Then the derived category of R -modules is

$$D(R) = \mathcal{Q}^{-1}K(R)$$

Proposition $D(R)$ exists.

Roughly, morphisms in $D(R)$ are equivalence classes of "fractions" $f s^{-1}: X \xleftarrow{s} X' \xrightarrow{f} Y$ where $s \in \mathcal{Q}$.
This creates a set-theoretic issue, since, for a given X , the class of quasi isomorphisms $X \xleftarrow{s} X'$ may not be a set.

Nevertheless, it is possible to model the equivalence classes of fractions by sets, and this is how $D(R)$ is constructed. This is why " $D(R)$ exists" is a proposition.

Prop $D(R)$ is a triangulated category, and $q: K(R) \rightarrow D(R)$ is an exact functor (commutes with shift, takes exact triangles to exact triangles.)

See Weibel Ch.10 for proofs.

At the end of the day, morphisms in $D(R)$ are related to something that we know from homological algebra.

Def Given $A, B \in Ch(R)$, define the hyperext groups
 $Ext^n(A, B) := Hom_{D(R)}(A, B[n])$

If $A, B \in mod-R$ are regarded as complexes in degree 0, $Ext^n(A, B)$ is the usual Ext group.

Note that for $A, B \in mod-R$,

$$Hom_{K(R)}(A, B[n]) = \begin{cases} Hom_{mod-R}(A, B), & n=0 \\ 0, & n \neq 0, \end{cases}$$

showing that $K(R)$ really differs from $D(R)$

Recall Ext: $A, B \in mod-R$. Replace A by a projective resolution (eg a free resolution)

$$\dots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} A \rightarrow 0$$

then compute $Hom_{mod-R}(P_i, B)$, take cohomology.

$\text{Ext}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2)$: Free resolution $0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}_2 \rightarrow 0$

$$\text{Hom}(\mathbb{Z}, \mathbb{Z}_2) \xleftarrow{0} \text{Hom}(\mathbb{Z}, \mathbb{Z}_2)$$

\parallel \parallel
 \mathbb{Z}_2 \mathbb{Z}_2

So $\text{Ext}_{\mathbb{Z}}^0(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2 = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2)$
 $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$, the non-identity element corresponds to the extension $0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \rightarrow 0$.

So $\text{Hom}_{D(\mathbb{R})}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$
 $\text{Hom}_{D(\mathbb{R})}(\mathbb{Z}_2, \mathbb{Z}_2[1]) = \mathbb{Z}_2$

The nontrivial morphism $\mathbb{Z}_2 \rightarrow \mathbb{Z}_2[1]$ is represented by the fraction $f s^{-1}$

