Lecture 4: [Finish previous notes, then:]

The derived category $D(R)$ is formed from the homotopy category $K(R)$ of cochain complexes of R-modules by "inverting quasi-isomorphisms." This is a process analogous to the localization of a ring (for instance, the field of fractions of an integral domain.)

Def Let S be a collection of morphisms in a category $\mathcal{C}$. A localization of $\mathcal{C}$ w.r.t. S is a category $S^{-1}\mathcal{C}$ and a functor $q: \mathcal{C} \to S^{-1}\mathcal{C}$ such that

1. $\forall s \in S$, $q(s)$ is an isomorphism in $S^{-1}\mathcal{C}$
2. If $F: \mathcal{C} \to \mathcal{D}$ is a functor such that $F(s)$ is an isomorphism in $\mathcal{D}$ for all $s \in S$, then $\exists!$ $\tilde{F}: S^{-1}\mathcal{C} \to \mathcal{D}$ such that $F = \tilde{F} \circ q$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "q"'] & \mathcal{D} \\ S^{-1}\mathcal{C} \arrow[ur, dashed, "\exists!\tilde{F}"'] & \end{tikzcd}$$

Because this is a universal property, $S^{-1}\mathcal{C}$ is unique if it exists.

Def Let $Q = \{ f: X \to X' \mid f \text{ is quasi-iso} \} \subset K(R)$.
Then the derived category of R-modules is

$$D(R) = Q^{-1}K(R)$$

Proposition $D(R)$ exists.

Roughly, morphisms in $D(R)$ are equivalence classes of "fractions" $fs^{-1}: X \xleftarrow{s} X' \xrightarrow{f} Y$ where $s \in Q$.
This creates a set-theoretic issue, since, for a given X, the class of quasi isomorphisms $X \xleftarrow{s} X'$ may not be a set.

Nevertheless, it is possible to model the equivalence classes of fractions by sets, and this is how $D(R)$ is constructed. This is why "$D(R)$ exists" is a proposition.

Prop $D(R)$ is a triangulated category, and $q: K(R) \to D(R)$ is an exact functor (commutes with shift, takes exact triangles to exact triangles.)

See Weibel Ch.10 for proofs.

At the end of the day, morphisms in $D(R)$ are related to something that we know from homological algebra.

Def Given $A, B \in Ch(R)$, define the hyperext groups

$$\mathrm{Ext}^n(A,B) := \mathrm{Hom}_{D(R)}(A, B[n])$$

If $A, B \in \text{mod-}R$ are regarded as complexes in degree 0, $\mathrm{Ext}^n(A,B)$ is the usual Ext group.

Note that for $A, B \in \text{mod-}R$,

$$\mathrm{Hom}_{K(R)}(A, B[n]) = \begin{cases} \mathrm{Hom}_{\text{mod-}R}(A,B), & n=0 \\ 0, & n \neq 0, \end{cases}$$

showing that $K(R)$ really differs from $D(R)$

Recall Ext: $A, B \in \text{mod-}R$. Replace A by a projective resolution (eg a free resolution)

$$\cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} A \to 0$$

then compute $\mathrm{Hom}_{\text{mod-}R}(P_i, B)$, take cohomology.

$\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2)$: Free resolution $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$

$$\begin{array}{ccc} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_2) & \xleftarrow{0} & \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_2) \\ \| & & \| \\ \mathbb{Z}_2 & & \mathbb{Z}_2 \end{array}$$

So $\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2 = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2)$
$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$, the nonidentity element corresponds to the extension $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$.

So $\mathrm{Hom}_{D(R)}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$
$\mathrm{Hom}_{D(R)}(\mathbb{Z}_2, \mathbb{Z}_2[1]) = \mathbb{Z}_2$

The nontrivial morphism $\mathbb{Z}_2 \to \mathbb{Z}_2[1]$ is represented by the fraction fs^{-1}

$$\begin{array}{ccccccccc} 0 & \to & 0 & \to & \mathbb{Z}_2 & \to & 0 \\ & & s\uparrow & & s\uparrow & & \\ 0 & \to & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \to & 0 \\ & & f\downarrow & & f\downarrow & & \\ 0 & \to & \mathbb{Z}_2 & \to & 0 & \to & 0 \end{array}$$

s is a quasi isomorphism