Start with Example on p.4 of previous notes.
Finish previous notes, then:

Sketch of proof that $K(R)$ is a triangulated category:

Axiom TR1 is clear from the construction: every map has a cone.

For axiom TR2, we may assume the exact triangle is of the form
$$A \xrightarrow{u} B \xrightarrow{v} \mathrm{Cone}(u) \xrightarrow{w} A[1]$$
We claim $\quad B \xrightarrow{v} \mathrm{Cone}(u) \xrightarrow{w} A[1] \xrightarrow{u[1]} B[1]$
which is to say that $\mathrm{Cone}(B \xrightarrow{v} \mathrm{Cone}(u))$ is chain homotopy equivalent to $A[1]$.

$$\mathrm{Cone}(u) = \left(B \oplus A[1],\ d_{\mathrm{cone}(u)} = \begin{pmatrix} d_B & u \\ & -d_A \end{pmatrix} \right)$$

$$\mathrm{Cone}(v) = \left((B \oplus A[1]) \oplus B[1],\ d = \left(\begin{array}{cc|c} d_B & u & 1_B \\ 0 & -d_A & 0 \\ \hline 0 & 0 & -d_B \end{array}\right) \right)$$

So $\mathrm{Cone}(v)$ contains a subcomplex $\left(B \oplus B[1],\ \begin{pmatrix} d_B & 1_B \\ 0 & -d_B \end{pmatrix} \right)$

which is clearly contractible: indeed $P = \begin{pmatrix} 0 & 0 \\ 1_B & 0 \end{pmatrix}$ is a map

such that $\quad dP + Pd = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} d & 1 \\ 0 & -d \end{pmatrix} + \begin{pmatrix} d & 1 \\ 0 & -d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$$= 1_{B \oplus B[1]}$$

Using this we can construct a homotopy equivalence
$$\mathrm{Cone}(V) \equiv \mathrm{Cone}(V) / B \oplus B[1] \cong A[1]$$

<u>Homework</u>: check the other rotation.

Axiom TR3 expresses the naturality of the mapping cone construction.

TR4: Now the dreaded octahedral axiom: let $A \xrightarrow{u} B \xrightarrow{v} C$ be maps
let $C' = \text{cone}(u)$, $A' = \text{cone}(v)$, $B' = \text{cone}(vu)$
The axiom asserts existence of maps $\left\{ \begin{matrix} f: C' \to B' \\ g: B' \to A' \end{matrix} \right\}$ with certain properties.

Now $C' = \left(B \oplus A[1], \begin{pmatrix} d_B & u \\ 0 & -d_A \end{pmatrix}\right)$ $\quad B' = \left(C \oplus A[1], \begin{pmatrix} d_C & vu \\ 0 & -d_A \end{pmatrix}\right)$

$$A' = \left(C \oplus B[1], \begin{pmatrix} d_C & v \\ & -d_B \end{pmatrix}\right)$$

Define $f: C' \to B'$ by $\quad f(b,a) = (v(b), a)$
$\qquad g: B' \to A'$ by $\quad g(c,a) = (c, u(a))$

Check: These are chain maps.

There is a natural map $A' \to \text{Cone}(f)$
since $\text{cone}(f) = C \oplus A[1] \oplus B[1] \oplus A[2]$
form A'

It remains to check all of the compatibilities in the octahedral diagram, in particular, that cone(f) is homotopy equivalent to A'. (omitted.) ∎

Given a cochain complex of R-modules $\cdots M^{i-1} \xrightarrow{d} M^i \xrightarrow{d} M^{i+1} \cdots$
we can take the i-th cohomology R-module

$$H^i(M) = \frac{\ker(d: M^i \to M^{i+1})}{\text{Im}(d: M^{i-1} \to M^i)}$$

By homotopy invariance of cohomology, $H^i: K(R) \to \text{mod-}R$ is a functor on the homotopy category.

The collection of functors $\{H^i\}_{i\in\mathbb{Z}}$ interact nicely with exact triangles:

Prop: If
$$\begin{tikzcd} & C \arrow[dl, "w"'] & \\ A \arrow[rr, "u"'] & & B \arrow[ul, "v"'] \end{tikzcd}$$
is an exact triangle in $K(R)$, then there is a long exact sequence in mod-R:

$$\cdots \to H^{i-1}(C) \xrightarrow{w^*} H^i(A) \xrightarrow{u^*} H^i(B) \xrightarrow{v^*} H^i(C) \xrightarrow{w^*} H^{i+1}(A) \to \cdots$$

<u>Rmk</u>: Axiomatizing this proposition as a property of $\{H^i\}$ yields the notion of a "cohomological functor" on a triangulated category.

For some purposes, $K(R)$ is not quite the right category for the homotopy theory of chain complexes: There are two related issues:

Def: A cochain complex $\{\cdots M^{i-1} \xrightarrow{d} M^i \xrightarrow{d} M^{i+1} \cdots\}$ is called <u>acyclic</u> if $H^i(M) = 0$ for all i.
A map $f: M^\bullet \to N^\bullet$ is called a <u>quasi-isomorphism</u> if the induced map on cohomology, $H^i(f): H^i(M) \to H^i(N)$ is an isomorphism for all i.
For example, a chain homotopy equivalence is always a quasi isomorphism.

The two (related) problems are: In $K(R)$,
1. There are quasi-isomorphisms that are not homotopy equivalences.
2. There are acyclic complexes that are not contractible.