Category-theoretic concepts

A <u>Category</u> $\mathcal{C}$ has objects $\mathrm{Ob}\,\mathcal{C}$ $\left(\begin{smallmatrix}\text{set or}\\ \text{class}\end{smallmatrix}\right)$ and morphisms $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ for $X,Y \in \mathrm{Ob}\,\mathcal{C}$.
There is a composition

$$\mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$$

and identity morphisms $1_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$ such that composition is associative and 1_X is an identity for composition

Let k be a commutative ring (eg. $k=\mathbb{Z}$ or k a field)

<u>Def</u> A <u>k-linear structure</u> on a category $\mathcal{C}$ consists of a k-module structure on each Hom set: $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a k-module $\left(\begin{smallmatrix}k\text{-vector space if}\\ k \text{ is a field}\end{smallmatrix}\right)$ such that composition is a linear operation.

i.e., given $A \xrightarrow{f} B \underset{g'}{\overset{g}{\rightrightarrows}} C \xrightarrow{h} D$ it holds that

$$h\circ(g+g')\circ f = hgf + hg'f \quad \text{and} \quad h(\lambda g)f = \lambda(hgf)$$
for $\lambda \in k$.

<u>Def</u> A <u>additive</u> k-linear category $\mathcal{C}$ is one which has a zero object (initial and terminal) and for any $X,Y \in \mathrm{Ob}\,\mathcal{C}$, a product object $X\times Y$.
These conditions imply that finite products and coproducts coincide, so we usually write $X \oplus Y$ for the (co)product.

<u>Example</u>: let R be an associative k-algebra
(R is an associative ring that contains k in its center)
let $\mathcal{C} = \text{mod}-R$, right R-modules, with module homs.
Then $\mathcal{C}$ is an additive k-linear category.
Given $f, f' \in \text{Hom}_R(M,N)$ define $(f+f')(m) = f(m) + f'(m)$

and given $\lambda \in k$ define $(\lambda f)(m) = f(m\lambda)$
and indeed $(\lambda f)(mr) = f(mr\lambda) = f(m\lambda r) = f(m\lambda)r = (\lambda f)(m)\cdot r$

since k central.

0 is the 0-module, $M \oplus N$ is the ordinary direct sum.

Traditionally, the next definition would be that of an Abelian category. We will skip this because it is less important for HMS than one might think.

<u>Def</u> A <u>k-linear triangulated category</u> $\mathcal{C}$ consists of a k-linear additive category with two more pieces of structure:

1. An automorphism $T : \mathcal{C} \to \mathcal{C}$, called the (autoequivalence) <u>translation</u> or <u>shift</u> functor.

Once T is specified, we have a notion of a <u>triangle of morphisms</u> this is a diagram of the form.

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX \quad \text{or} \quad \begin{tikzcd} & Z \arrow[dl, "w"', "T"] & \\ X \arrow[rr, "u"'] & & Y \arrow[ul, "v"'] \end{tikzcd}$$

2. A class of <u>distinguished triangles</u> in $\mathcal{C}$.
(also called exact triangles)

These structures are required to satisfy Axioms (TR1–4)

(TR1) (a) Given any morphism $u: X \to Y$, there is a distinguished triangle containing it: $\exists\, Z$ and $v: Y \to Z$, $w: Z \to TX$ such that (u, v, w) is distinguished.

(b) The triangle $X \xrightarrow{Id} X \xrightarrow{0} 0 \xrightarrow{0} TX$ is distinguished

(c) If
$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "\cong"'] & Y \arrow[r, "v"] \arrow[d, "\cong"'] & Z \arrow[r, "w"] \arrow[d, "\cong"'] & TX \arrow[d, "\cong"'] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX' \end{tikzcd}$$
is commutative and verticals are isos, then top row is distinguished iff bottom row is.

(TR2) If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ is distinguished, so are
$$T^{-1}Z \xrightarrow{-T^{-1}w} X \xrightarrow{u} Y \xrightarrow{v} Z \quad \text{and} \quad Y \xrightarrow{v} Z \xrightarrow{w} TX \xrightarrow{-Tu} TY$$
"Can rotate triangles 120° degrees"

(TR3) Given
$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] & TX \arrow[d, "Tf"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX' \end{tikzcd}$$
rows distinguished Δ's and diagram commutative, then $\exists\, h: Z \to Z'$ making diagram commute

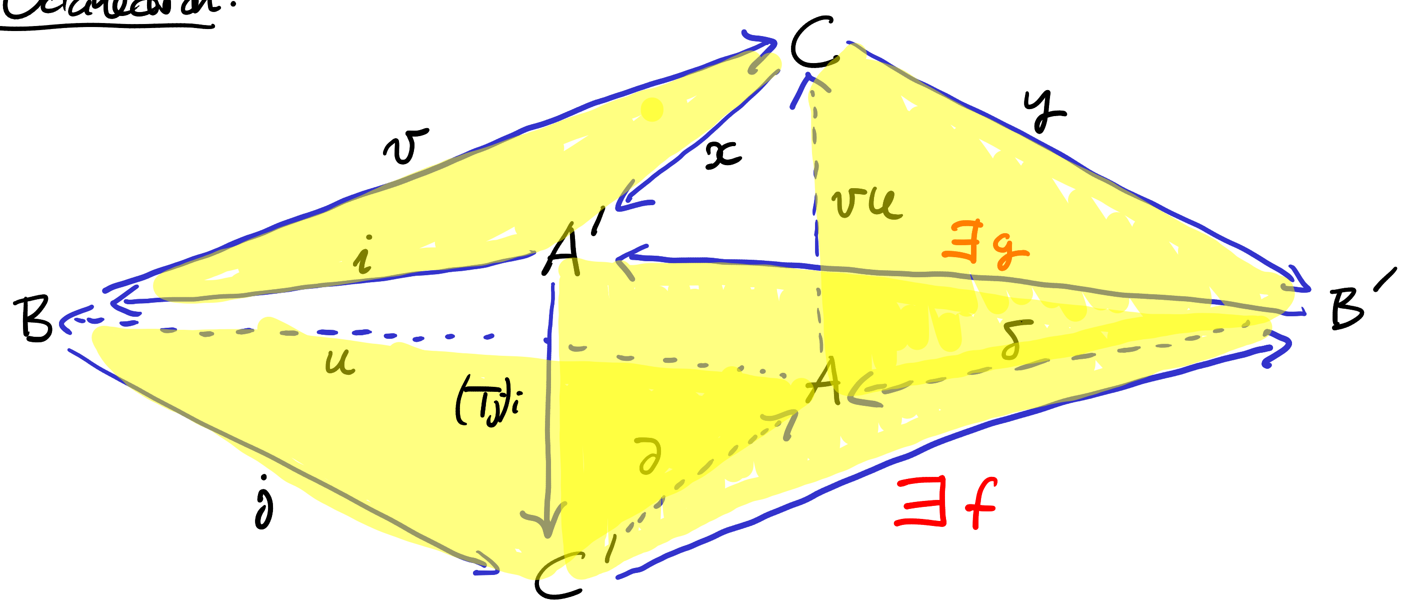
(TR4) (The dreaded octahedral axiom, essentially about the cone of a composition)

Suppose $A \xrightarrow{u} B \xrightarrow{v} C$ are any pair of composable morphisms.

Suppose
$$\left.\begin{array}{l} A \xrightarrow{u} B \xrightarrow{j} C' \xrightarrow{\partial} TA \\ B \xrightarrow{v} C \xrightarrow{x} A' \xrightarrow{i} TB \\ A \xrightarrow{vu} C \xrightarrow{y} B' \xrightarrow{\delta} TA \end{array}\right\} \text{ are distinguished}$$

Then there is a distinguish triangle $C' \xrightarrow{f} B' \xrightarrow{g} A' \xrightarrow{(Tj)i} TC'$ where $\partial = \delta f$, $x = gy$, $yv = fj$ and $u\delta = ig$

Octahedron:



Shaded faces are exact, other faces commute.

Example: Start with mod-R. Then form $Ch(R)$, the category of cochain complexes of R-modules

Objects = complexes $\{\cdots M^{i-1} \xrightarrow{d_{i-1}} M^i \xrightarrow{d_i} M^{i+1} \xrightarrow{d_{i+1}} M^{i} \cdots\}$

M^i R-modules, d_i R-module maps
$d_{i+1} \circ d_i = 0$.

Morphisms = cochain maps

$$\begin{tikzcd} \cdots M^{i-1} \arrow[r] \arrow[d, "f_{i-1}"'] & M^i \arrow[r, "d_i^M"] \arrow[d, "f_i"] & M^{i+1} \cdots \arrow[d, "f_{i+1}"] \\ \cdots N^{i-1} \arrow[r] & N^i \arrow[r, "d_i^N"] & N^{i+1} \cdots \end{tikzcd}$$

$$f_{i+1} \circ d_i^M = d_i^N \circ f_i$$

The translation $T : Ch(R) \to Ch(R)$ is the shift in the indexing $(TM)^i = M^{i+1}$ this is also denoted $M[1]$ with differential $-d$.

The cone of a morphism $M \xrightarrow{f} N$ is

$$\mathrm{Cone}(f) = N \oplus M[1]$$

So $(\mathrm{Cone}(f))^i = N^i \oplus M^{i+1}$ $\quad (\mathrm{Cone}(f))^{i+1} = N^{i+1} \oplus M^{i+2}$

the differential $d^{\mathrm{cone}(f)} : N^i \oplus M^{i+1} \to N^{i+1} \oplus M^{i+2}$

is

$$d^{\mathrm{cone}(f)} \begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} d_i^N & f \\ 0 & -d_{i+1}^M \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix}$$

$$(d^{\mathrm{cone}(f)})^2 = \begin{pmatrix} d^N & f \\ 0 & -d^M \end{pmatrix} \begin{pmatrix} d^N & f \\ 0 & -d^M \end{pmatrix} = \begin{pmatrix} (d^N)^2 & d^N f - f d^M \\ 0 & (d^M)^2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$ since d^M and d^N are differentials and f is a cochain map.

<u>Def</u> Two cochain maps $f, f' : M \to N$ are <u>homotopic</u> if $\exists\ P_i : M^i \to N^{i-1}$ such that

$$f_i' - f_i = d_{i-1}^N P_i + P_{i+1} d_i^N$$

The <u>homotopy category</u> $K(R)$ of cochain complexes of mod-R is the category with the same objects as $Ch(R)$ but with homotopic morphisms identified

$$\mathrm{Hom}_{K(R)}(M,N) = \mathrm{Hom}_{Ch(R)}(M,N) / \simeq$$

<u>Theorem:</u> Letting the distinguished triangles be those that are isomorphic to $M \xrightarrow{f} N \to \mathrm{Cone}(f) \to M[1]$, $K(R)$ is a triangulated category.