Math 595 Homological Mirror Symmetry

Two geometries

(A) Symplectic manifold (M, ω)

M smooth manifold of dimension $2n$

$\omega \in \Omega^2(M)$, $d\omega = 0$, $\omega^n > 0$.

(B) Algebraic variety X

e.g. compact complex submanifold of $\mathbb{CP}^n$

or more abstractly, a scheme.

In homological mirror symmetry (HMS)

Geometries (A) and (B) meet in the form of triangulated categories.

$$\begin{array}{ccccc}
\text{Symplectic manifolds} & \longrightarrow & \text{Triangulated categories} & \longleftarrow & \text{Algebraic varieties} \\
(M,\omega) & \longmapsto & \mathcal{F}(M,\omega) \text{ "Fukaya Category"} & & \\
 & & \text{Derived category } D^b(X) & \longleftarrow\!\shortmid & X
\end{array}$$

When $\mathcal{F}(M,\omega)$ and $D^b(X)$ are "equivalent", we say (M,ω) and X are "HMS partners" and that "HMS holds" for this pair.

There is a lot of motivation and history behind this idea, but for now, let's have a look at a precise theorem that we can (hope to) prove this semester.

<u>Note</u>: Some of the concepts in what follows are likely new to you. In this course we will provide complete definitions. (just not all today.)

Seidel's Quartic Surface Theorem

<u>Coefficients</u>: let $\Lambda_{\mathbb{N}} = \mathbb{C}[[q]]$ be ring of formal powerseries
let $\Lambda_{\mathbb{Z}} = \mathbb{C}((q))$ be formal Laurent series

let $\Lambda_{\mathbb{Q}}$ = algebraic closure of $\Lambda_{\mathbb{Z}}$

$\Lambda_{\mathbb{Q}}$ consists of Puiseux series $f = \sum_{m \in \mathbb{Q}} a_m q^m$

such that $a_m = 0$ for $m \ll 0$ and all m such that $a_m \neq 0$ can be taken to have a common denominator.

· The q-adically continuous Galois group of $\Lambda_{\mathbb{Z}} / \mathbb{C}$ consists of formal changes of variable
$$\psi^* : q \mapsto \psi(q) \quad \text{for } \psi \in q\,\mathbb{C}[[q]]$$

<u>Symplectic side</u> let $\mathbb{CP}^3$ be the complex projective space
let $M_p \subseteq \mathbb{CP}^3$ be a submanifold defined by the vanishing of a quartic homogeneous polynomial p
e.g. $p(x_0, x_1, x_2, x_3) = x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$

$\mathbb{CP}^3$ carries a symplectic Fubini-Study form ω_{FS}
let $\omega_p = \omega_{FS}|_{M_p}$; this is a symplectic form on M_p

<u>lemma</u>: Up to symplectic diffeomorphism, (M_p, ω_p) does not depend on the choice of the quartic polynomial p
(All that matters is that 0 be a regular value of p so that M_p is a smooth manifold)

To (M_p, ω_p) we associate the split-closed derived Fukaya category $D^\pi \mathcal{F}(M_p, \omega_p)$
which is a $\Lambda_{\mathbb{Q}}$-linear triangulated category
(Hom sets are $\Lambda_{\mathbb{Q}}$-vector spaces)
Defined using (rational) Lagrangian submanifolds of M_p and pseudo-holomorphic curves.

Algebraic side: Inside $\mathbb{P}^3_{\Lambda_{\mathbb{Q}}}$ consider the quartic surface

$$Y = \{ y_0 y_1 y_2 y_3 + q(y_0^4 + y_1^4 + y_2^4 + y_3^4) = 0 \} \subset \mathbb{P}^3_{\Lambda_{\mathbb{Q}}}$$

There is a group action by $\Gamma_{16} \cong \mathbb{Z}_4 \times \mathbb{Z}_4$

$$\Gamma_{16} = \left\{ \left[\begin{pmatrix} \alpha_0 & & & \\ & \alpha_1 & & \\ & & \alpha_2 & \\ & & & \alpha_3 \end{pmatrix} \right] \;\middle|\; \begin{array}{l} \alpha_k^4 = 1, \\ \alpha_0\alpha_1\alpha_2\alpha_3 = 1 \end{array} \right\} \subset PSL(4, \Lambda_{\mathbb{Q}})$$

by scaling the coordinates.

Denote by Z_q^* the unique minimal surface resolving the singularities of Y/Γ_{16}.
(Since $\Lambda_{\mathbb{Q}}$ is an algebraically closed field of characteristic zero, this is classical.)

To Z_q^* we associate the Bounded derived category of coherent sheaves $D^b Coh(Z_q^*)$
Also a $\Lambda_{\mathbb{Q}}$-linear triangulated category

Theorem (Seidel) There is a $\psi \in q\mathbb{C}[[q]]$ and an equivalence of categories

$$D^\pi \mathcal{F}(M_p, \omega_p) \simeq \hat{\psi}^* D^b Coh(Z_q^*)$$

Here $\hat{\psi}$ is any lift of $\psi \in Gal_q(\Lambda_{\mathbb{Z}}/\mathbb{C})$ to $Gal_q(\Lambda_{\mathbb{Q}}/\mathbb{C})$ and $\hat{\psi}^* D^b Coh(Z_q^\times)$ means we twist the $\Lambda_{\mathbb{Q}}$-linear structure by this automorphism.
[ψ corresponds to the "mirror map" between moduli spaces]

What are triangulated categories, and how do they arise from geometry?

Homological Algebra: let R be an associative ring, and let mod-R denote the category of right R-modules.
Objects = right R-modules
Morphisms = R-module homomorphisms.

mod-R has more structure than just a category, because we can do things like add objects ($A \oplus B$) and morphisms we can take kernels of morphisms, we have exact sequences, and so on.

We can look at cochain complexes of R-modules

$$\cdots \to M_{i-1} \xrightarrow{d_{i-1}} M_i \xrightarrow{d_i} M_{i+1} \xrightarrow{d_{i+1}} M_{i+2} \to \cdots$$

each M_i an R-module, d_i's module maps
$(\forall i)$ $d_i \circ d_{i-1} = 0$

Cohomology modules $H^i(M) = \dfrac{\ker d_i}{\operatorname{Im} d_{i-1}}$

There is a category $Ch(\text{mod-}R)$ of cochain complexes and cochain maps $f: M \to N$

$$\begin{array}{ccc} M_i & \xrightarrow{d_i} & M_{i+1} \\ f_i \downarrow & & \downarrow f_{i+1} \\ N_i & \xrightarrow{d_i} & N_{i+1} \end{array}$$

Then we form $K(\text{mod-}R)$ = homotopy category of $Ch(\text{mod-}R)$
same objects but morphisms are
homotopy classes of cochain maps.

Then we form $D(\text{mod-}R)$ from $K(\text{mod-}R)$
by taking the "Verdier quotient" by the subcategory
of acyclic complexes (ones for which $H^i(M)=0\ \forall i$)

This $D(\text{mod-}R)$ is called the Derived category of R-modules
and it is the right setting for the "homotopy theory"
of R-modules.

The notion of "Triangulated category" is the abstraction
of the salient features of $D(\text{mod-}R)$