Bohan Fang: Givental Quantization

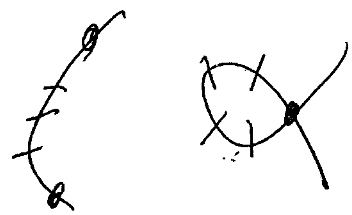
I. CohFT (Kontsevich-Manin)

- V vector space w/ non-deg pairing
- $I_{g,n}: V^{\otimes n} \longrightarrow H^*(\overline{\mathcal{M}}_{g,n})$

1) $\varphi_S: \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$

$$\varphi^*(I_{g,n}(\gamma_1 \otimes \cdots \otimes \gamma_n)) = \pm \sum_{a,b} I_{g_1,n_1+1}\Big(\bigotimes_{j\in S_1}\gamma_j \otimes H_a\Big) g^{ab} I_{g_2,n_2+1}\Big(H_b \otimes \bigotimes_{j\in S_2}\gamma_j\Big)$$

2) $\varphi: \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$



$$\varphi^*(I_{g,n}(\gamma_1 \otimes \cdots \otimes \gamma_n)) = \sum_{a,b} I_{g-1,n+2}(\gamma_1 \otimes \cdots \otimes \gamma_n \otimes H_a \otimes H_b) \cdot g^{ab}$$

$\gamma_1, \dots, \gamma_n \in V$ $\quad \langle \gamma_1, \dots, \gamma_n \rangle_{g,n} \overset{\text{def}}{=} \int_{\overline{\mathcal{M}}_{g,n}} I_{g,n}(\gamma_1 \otimes \cdots \otimes \gamma_n)$

X cpt Kähler, <u>Def</u> $\langle I_{g,n,\beta}(\gamma_1, \dots, \gamma_n) \cup \alpha, [\overline{\mathcal{M}}_{g,n}] \rangle$

$$= \langle \alpha \cup ev_1^*(\gamma_1) \cdots ev_n^*(\gamma_n), [\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} \rangle$$

$$I_{g,n} = \sum_{\beta \geq 0} I_{g,n,\beta}$$

II. Frobenius manifold $g=0$, $n=3$ data

Algebra: $(V, *)$ $*$ unital assoc. V has $(\ ,\ )$

Require semisimplicity: $\exists$ canonical basis $\{\phi_\alpha\}$

$$\phi_\alpha * \phi_\beta = \delta_{\alpha\beta}\phi_\alpha$$

Frobenius manifold: $\mathbb{C}$-manifold flat metric local function F

$$F_{abc}\big|_{\tau} = (\partial_a *_\tau \partial_b, \partial_c)$$

gives Frobenius algebra structure on tangent bundle

- $\mathbb{1}$ vector field cov. constant v.f of the flow preserves $*_\tau$

Generically semisimple: Tangent space algebra is generically semisimple.

CohFT (given) $\langle\langle \gamma_1, \dots, \gamma_n \rangle\rangle_{g,n}$ ← fnctn of $\tau \in V$

$$= \sum_{l\geq 0} \int_{[\overline{M}_{g,n+l}]} I_{g,n}(\gamma_1 \otimes \cdots \otimes \gamma_n \otimes \tau \otimes \cdots \otimes \tau)$$

Given CohFT $(V, I_{g,n})$ define

$$(\alpha * \beta, \gamma) \stackrel{def}{=} \langle\langle \alpha, \beta, \gamma \rangle\rangle_{0,3}$$

PDE: $H_1 \cdots H_N$ flat basis. $\tau_1 \cdots \tau_N$ flat coordinate

$$\frac{\partial}{\partial \tau_i} = H_i$$

$$\nabla_a = z\partial_a - H_a *_\tau$$

$\nabla \eta = 0$ solution. Choose a basis of solutions ~~(correct basis)~~

reconstruct higher genus

"CohFT" ↖ maybe different choices.

Ex: Toric (e.g. compact) smooth $X \circlearrowleft T$

$$V = H^*_T(X)$$

Frobenius manifold V $\quad T_\tau V = V$

$\phi^\alpha = [p_\alpha]$ $\quad p_1, \dots, p_N$ torus fixed pt.

$$\phi_\alpha = \frac{\phi^\alpha}{e_T(T_{p_\alpha} X)} \qquad \phi_\alpha \cup \phi_\beta = \delta_{\alpha\beta}\phi_\alpha$$

classically semisimple

quantum product $*_\tau \to \cup$

$$\begin{array}{c} \tau = \tau' + \tau'' \\ \in H^2 \quad \in H^{\neq 2} \end{array} \quad \Bigg| \quad \begin{array}{c} \text{as } \tau' \to -\infty \\ \tau'' \to 0 \\ \text{Large radius limit} \end{array}$$

ϕ_α canonical basis at $\tau = LRL$

ϕ^α $\quad \phi^{\alpha(\tau)}$ $\phi_{\alpha(\tau)}$ canonical basis around $\tau = LRL$

$$(\phi_\alpha(\tau), \phi_\alpha(\tau)) = \frac{1}{\Delta_\alpha(\tau)}$$

$$(\phi_\alpha, \phi_\alpha) = \frac{1}{\Delta_\alpha} \qquad \hat{\phi}_\alpha(\tau) \text{ normalized.}$$

$(\eta_1, \dots, \eta_N)$ fundamental solutions.

$$\eta_a(\tau) = \sum_{a=1}^{N} (S_\tau)_a{}^\alpha \hat{\phi}_\alpha$$

$(S_\tau)_a{}^\alpha$ satisfies QDE.

Dubrovin - Givental:

$$(S_\tau)_a{}^\alpha = (\Psi_\tau)_a{}^\beta R_\tau(z)_\beta{}^\alpha e^{u^\alpha(\tau)/z}$$

Choose flat basis H_a (Ψ_τ)

$$H_a = \sum_\beta (\Psi_\tau)_a{}^\beta \hat{\phi}_\beta(\tau)$$

$u^\alpha(t)$ canonical coordinate $\frac{\partial}{\partial u^\alpha} = \phi_\alpha(\tau)$

$R(z)_\beta{}^\alpha$ power series in z. But $e^{u_\alpha(\tau)/z}$ is series in z^{-1}!

$$\rightsquigarrow (d + W\wedge) R_{k-1} = [dU, R_k]$$

$$R = \mathrm{id} + R_1 z + R_2 z^2 + \cdots$$

$$W = \Psi_\tau^{-1} d\Psi_\tau \qquad U = \mathrm{diag}(u^\alpha).$$

Conditions for $R_\tau(z)^\alpha_\rho$

Thm (Givental)

0) solution exist.

1) R unitary $\sum_\gamma R_\alpha{}^\gamma(z) R_\gamma{}^\beta(-z) = \delta_{\alpha\beta}$

2) R unique up to

$\times\ e^{\sum_i a_{2i-1} z^{2i-1}}$ $\quad a_{2i-1}$ = diagonal

Givental says given $R \to$ get a CohFT.

Teleman Any semisimple CohFT comes from givental's construction.

CohFT: $\mathbf{t}(z) = t_0 + t_1 z + \cdots \quad t_i \in V$

$$F^g_{\mathbf{t}} = \sum_{n=0}^{\infty} \frac{1}{n!} \langle\langle \mathbf{t}(\bar\psi_1), \ldots, \mathbf{t}(\bar\psi_n) \rangle\rangle^V_{g,n}$$

cotangent lines on $\overline{M}_{g,n}$

$$A_I(\mathbf{t}) = \sum_{g \geq 0} \hbar^{g-1} F^g_{\mathbf{t}}$$

$$\hat{A}_I(\mathbf{t}) = \hat{\Psi}\, \hat{R}(z) \prod_{\alpha=1}^{N} T(\mathbf{t}^\alpha)$$

Trivial CohFT

$$\mathbf{t}(z) = \sum \mathbf{t}^\alpha(z)\, \hat{e}_\alpha(\tau)$$

$$T(\mathbf{t}^\alpha) = \exp\left(\sum_{g=0}^{\infty} \frac{\hbar^{g-1}}{n!} \langle \mathbf{t}^\alpha(\bar\psi_1), \ldots, \mathbf{t}^\alpha(\bar\psi_n) \rangle^{pt}_{g,n}\right)$$

intersection theory on $\overline{M}_{g,n}$

$\hat{A}_I(\mathbf{t})$ is a CohFT; Teleman: Any CohFT appears this way.