Andrei Caldararu Higher genus B-model.

Costello: Categorical Mirror Symmetry (joint w/ Junwu Tu)

$\Downarrow$

"Classical Mirror Symmetry"

Classical computation (Dijkgraaf)

$E \rightsquigarrow$ target (elliptic curve)

$$\begin{pmatrix} k & \ell \\ 0 & d/k \end{pmatrix} \quad 0 \le \ell < k$$

$$F^A_{1,1}(q) = -\frac{1}{24} + \sum_{d\geq 1}\Big(\sum_{k|d} k\Big)q^d = -\frac{1}{24}E_2(q)$$

\# of isogenies of degree d.

compute the Gromov-Witten invariant by inserting $[pt]^{PD} \in H^2(E, \mathbb{Q})$

Eisenstein quasi-Modular form. (but holomorphic)

Mirror symmetry:

Mirror map $\tau \longmapsto q = e^{2\pi i \tau}$

complex modulus of mirror $E_\tau = \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau$

$\rightsquigarrow$ From $D^b(E_\tau)$ + some data, should compute $F^B_{1,1}(\tau) = F^A_{1,1}(q)$

Costello: compute the string vertex at λ^1 $\left(l = -\chi\left(\substack{\text{surface} \\ \text{of } g=1,\, n=1}\right)\right)$ 2
can do this once and for all.

Find a CY A_∞-algebra equiv to $D^b(E_\tau)$
A_τ

Find analogue of $[pt]^{PD}$ + its lift to cyclic homology.

$HH_{-1}(A_\tau) \leftarrow HC(A_\tau)$

Kontsevich-Soibelman insertions in Ribbon graphs
$F^B_{1,1}(\tau)$

① The String vertex (Zwiebach - Sen) (Background independent)

$$\mathcal{F} = \bigoplus_{g,n} C_*\left(\mathcal{M}^{\text{discon}}_{g,n}/\Sigma_n\right)[[\lambda]]$$ is a BV-algebra.

product = disjoint union.

BV operator = sew $\circ \frown \circ$ $\xrightarrow{\text{with}}$ all possible rotation.

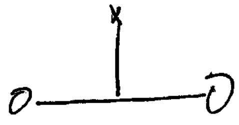
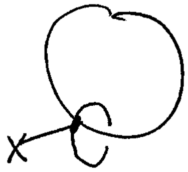
Take

Thm: (Costello) There exists a unique up to homotopy solution to $(d+\Delta)e^{S'}=0$, $S_{0,3}=\frac{1}{6}[pt]$

$$S' = \sum_{g,n} S_{g,n}\lambda^{2-2g-n} \qquad S_{g,n}\in C_*(\mathcal{M}^{disc}_{g,n}/\Sigma_n)$$

Quantum Master equation.

The coefficient of λ^1 in S

$\frac{1}{2}$  $+\frac{1}{4}$  $+\frac{1}{24}$ 

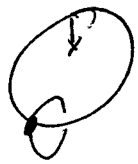
cyclic in variable

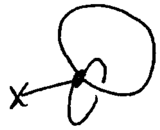
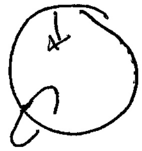
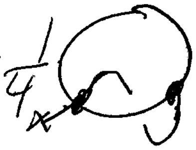
$C_0(\mathcal{M}_{0,3})$ 1 in 2 out

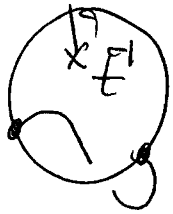
$C_2(\mathcal{M}_{1,1})$ 1 in 0 out

$\underbrace{C_0(\mathcal{M}_{1,1})t^{-1}}_{\deg 2}$ 1 in 0 out.

after turning an out put into an input.

$\Delta\left(\frac{1}{2}\right.$  $\left.\right) = \frac{1}{2}$ 

$d\left(\frac{1}{4}\right.$  $\left.\right) = \frac{2}{4}$  $+\frac{1}{4}$ 

$(d+tD)$ 

rotates distinguished vertex

2) Polishchuk: $\mathcal{O}\oplus L$ on E_τ

L = line bundle of degree 1

$$A_\tau = Ext^*(\mathcal{O}\oplus L, \mathcal{O}\oplus L) \qquad \text{6 dimensional}$$

$$\begin{tikzcd} \mathcal{O} \arrow[loop left, "\xi\ \deg 1"] \arrow[r, bend left, "\theta\ \deg 0"] & L \arrow[l, bend left, "\eta\ \deg 1"] \arrow[loop right, "\xi_L\ \deg 1"] \end{tikzcd}$$

Higher multiplication $m_{a+b+c+d+3}(\xi^a, \theta, \xi_L^b, \eta, \xi^c, \theta, \xi_L^d)$

$$= M(a,b,c,d)(\tau)\,\theta$$

↑ modular, but quasi holomorphic form.

$\mathbb{Q}[E_2, E_4, E_6]$ all holomorphic and quasi-modular, $\partial/\partial\tau$

$\varphi \downarrow$ Kaneko-Zagier

$\mathbb{Q}[E_2^*, E_4, E_6]$ $\qquad \partial/\partial\tau + \frac{K}{\tau-\bar\tau}$

<u>Thm</u> Apply the Kaneko-Zagier isomorphism to the structure constants of A_τ produces an A_∞-alg quasiequivalent to it.

(Get holomorphic dependence on τ) (Cf. Shuldam Renate)

③ Find the correct lift to insertion.

$$[pt]^{P.D.} \longrightarrow \underbrace{\frac{1}{2\pi i}\,\frac{\pi}{\mathrm{Im}\,\tau}\,d\bar z}_{\tilde\xi} \in HH_{-1}(A_\tau)$$

$$= \frac{1}{2\pi i}\xi \qquad \xi \in A_\tau^{\otimes 1}$$

The lift: On E_τ

$$0 \to H^0(\Omega^1) \to H^1_{dR}(E_\tau) \to H^1(\mathcal{O}) \to 0$$

$$\begin{array}{ccccc} \Psi & & & & \Psi \\ dz & & ? & \longleftarrow & \xi = \frac{\pi}{\operatorname{Im}\tau} d\bar{z} \\ \text{ambiguity} & & & & \end{array}$$

Mirror symmetry tells you you want the lift to be invariant under monodromy around the cusp.

Thm: TFAE: ① The lift is invt under monodromy.

② The lift is G-M flat

③ $KS(\partial_{\bar{\tau}}) \cup - = 0$ ← easiest to check.

Final computation: Fix τ ⇝ compute $M(a,b,c,d)(\tau)$ and $\frac{\partial}{\partial\tau} M(a,b,c,d)|_\tau$

Write code: matrices for $A^{\otimes 9} \oplus A^{\otimes 7} \oplus \cdots \oplus A^{\otimes 3} \xrightarrow{b} A^{\otimes 8} \oplus \cdots \oplus A^{\otimes 2}$

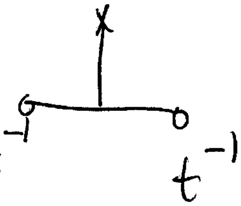
$$A^{\otimes 9} \oplus \cdots \oplus A^{\otimes 3} \xrightarrow{b^{1|1}(v_{\mu^*}, -)} A^{\otimes 7} \oplus \cdots \oplus A^{\otimes 1}$$

$KS(\partial_\tau) \cup -$ $\qquad \tilde{\xi} = \xi + t\alpha + \text{h.o.t.}$

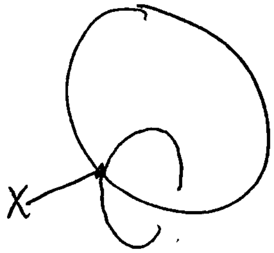
Solve linear system $b(\alpha) = 1 \otimes \xi$ ← $\tilde{\xi}$ is a lift of ξ

$b^{1|1}(v_{\mu^*}, \alpha) = 0$ ← $KS(\partial_\tau) \cup \tilde{\xi} = 0$.

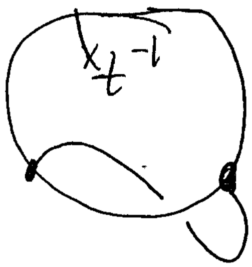
This is the correct lift.

Do: Insert ξ in $\frac{1}{2}$ 

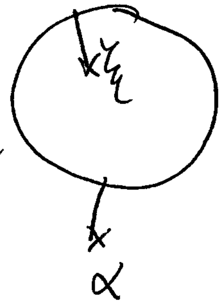
$\rightsquigarrow \frac{1}{2}(\xi t^{-1})(\xi t^{-1})$

Insert ξ 

$\rightsquigarrow 0$

Insert $t\alpha$ 

$\rightsquigarrow 0$

Compute $\frac{1}{2}\langle \alpha, \xi \rangle_{\text{Mukai}} = \frac{1}{2}$ 

$= \frac{1}{2} e_2(\tau)$

$$\rightsquigarrow \frac{1}{(2\pi i)^2} \cdot \frac{1}{2} e_2(\tau) = -\frac{1}{24} E_2(\tau)$$

Where does Mukai pairing come in

$H = HH_*(A)$

$H((t))$ $\qquad \langle \alpha t^k, \beta t^l \rangle = \langle \alpha, \beta \rangle \operatorname{Res} t^k (-t)^l$

$\cup$

$H[[t]]$ Lagrangian.

$$\mathrm{Weyl}(H((t))) \rightsquigarrow \mathcal{F}_H := \frac{\mathrm{Weyl}(H((t)))}{\langle H[[t]] \rangle}[[\lambda]]$$

$$\simeq \mathrm{Sym}(t^{-1}H[t^{-1}])[[\lambda]]$$

$\Downarrow$

GW potential

Deformed Fock-module:

$H[[t]] \xrightarrow[\text{unkeeding}]{\text{(sometimes}} V[[t]]$

Before: $xt^k = 0 \qquad k \geq 0$

$V = CC_*(A)$

Now mod by (xt^k inserted in $e^{S'}$) $\mathcal{F}_D$

Thm $\mathcal{F}_D/(\lambda) \simeq \mathcal{F}_H/(\lambda)$

$\mathcal{F}_D$ is flat $/ k[[\lambda]]$ $\Big\} \Rightarrow \mathcal{F}_D \simeq \mathcal{F}_H$

$1 \to \mathcal{I}$

Want result of inserting $[\xi] \in HH_{-1}$
at $g=1$, $n=1$

want coeff of $[\xi]^\vee \lambda^1 t^{-1}$ in $\mathbf{F}$.
Mukai dual

In $\mathcal{F}_H$ you read this by reading coeff of λ after acting by $[\xi]$ on $\mathbf{F}$

$$\begin{tikzcd} & \mathcal{W} \arrow[dl, "\text{acts}"'] \arrow[dr] & \\ \mathcal{F}_D \arrow[rr, "\varphi", "\sim"'] & & \mathcal{F}_H \end{tikzcd}$$

$$\mathbb{1} \longmapsto F$$

Act by $[\xi]$ on $\mathbb{1} \in \mathcal{F}_D$ then apply φ

Lift ξ to $\xi + t\alpha + O(t^2)$ insert into e^S.

get: $(\xi t^{-1})^2_\lambda \mathbb{1} \xrightarrow{\varphi} ?$

Guess: Maybe $[\xi t^{-1}]^2 \mathbb{1}_H$ is the image of $[\xi t^{-1}]^2 \mathbb{1}_D$
$([\xi] t^{-1})^2$ acting on $\mathbb{1}_D$? $[\xi] t^{-1}$ lifts to $\xi t^{-1} + \alpha$

$$(\xi t^{-1} + \alpha)(\xi t^{-1} + \alpha)\mathbb{1}_D = (\xi t^{-1})^2 \mathbb{1}_D + \cancel{\alpha^2 \mathbb{1}_D} + \cancel{\xi t^{-1} \alpha \mathbb{1}_D}$$
$$+ \alpha \xi t^{-1} \mathbb{1}$$

Need to subtract this pairing. $\xrightarrow{\text{Weyl relation}}$ $\langle \alpha, \xi \rangle + \cancel{\xi t^{-1}} \alpha \mathbb{1}$