Si Li   BCOV theory and Quantum B-model

Outline: ① BCOV theory and period map.

② Quantization

③ Elliptic curve example

X: CY geometry (compact CY/LG)

Ω_X: CY volume form.

$$PV(X) = \Omega^{0,\bullet}(\Lambda^\bullet T_X^{1,0}) = \bigoplus_{i,j} PV^{i,j}$$

$$PV^{i,j} = \Omega^{0,j}(\Lambda^i T_X)$$

$$PV^{i,j} \overset{\lrcorner \Omega_X}{\longleftrightarrow} \Omega^{d-i,j}$$

$$\bar\partial, \partial \longleftrightarrow \bar\partial, \partial \qquad \partial : PV^{i,j} \to PV^{i-1,j} \text{ divergence w/r/t } \Omega_X$$

- $\{\alpha, \beta\} = \partial(\alpha\beta) - (\partial\alpha)\beta - (-1)^\alpha \alpha\partial\beta$

- $Q = \bar\partial + t\partial$

Period map: Assume X CY3

Consider $\mathrm{Def}(X, \Omega_X) \longrightarrow H^3(X_0, \mathbb{C})$

$$[X, \Omega_X] \longrightarrow [\Omega_X]$$

- $H^3(X, \mathbb{C})$ is a symplectic space $\int_X \alpha \wedge \beta$

- $\mathrm{Im}\, P$ is a Lagrangian (Linear)

- Let $L \subset H^3$ be Lagrangian s.t.

$$H^3 = F^2 \oplus L \qquad H^3 \cong T^* F^2$$

$$F^2 \cong (H^{3,0} \oplus H^{2,1})$$

$$\mathrm{Im}(P_X) = \mathrm{Graph}(d\mathcal{F}_0)$$

$$\mathcal{F}_0 \in \mathcal{O}(H^{3,0} \oplus H^{2,1})$$

"mirror to GW at $g=0$"

Diff L leads to diff theory.

Remark: $\mathrm{Def}(X, \Omega_X)$ $\mu \in PV^{1,1}$ $\bar\partial \mu + \frac{1}{2}\{\mu, \mu\} = 0$

$$d\left(e^{\mu} \lrcorner \Omega_X\right) = 0 \qquad \text{CY volume form}$$

$$\Longleftrightarrow \quad Q(\mu + t\rho) + \frac{1}{2}\{\mu + t\rho, \mu + t\rho\} = 0$$

$$Q = \bar\partial + t\partial \qquad t \text{ formal variable}$$

"cyclic cohomology describes def(X, Ω_X)

<u>Def^n</u>: The fields of BCOV theory

$$\left(\mathcal{S}_+ = PV(X)[[t]],\quad Q = \bar\partial + t\partial \right)$$

$$\{\ ,\ \}$$

Let
$$\mathcal{S}_- = t^{-1}PV(X)[t^{-1}]$$
$$\mathcal{S} = PV(X)((t))$$

$$\Gamma: (\mathcal{S}, Q) \longrightarrow \Omega^\bullet(X)((t))$$

$$t^k\alpha \longrightarrow t^{k+i-1}\alpha \lrcorner \Omega_X,\quad \alpha \in PV^{i,*}$$

$$Q \longrightarrow d$$

$$\Gamma(\mathcal{S}_+) = \prod_{p\in\mathbb{Z}} t^{d-p-1}\,\Omega^{\geq p,*}$$

$\mathcal{S}_+ \subset \mathcal{S} \Longleftrightarrow$ Hodge filtration.

Symplectic Pairing on $\mathcal{S}$

$$\omega(f(t)\alpha, g(t)\beta) = \left(\operatorname{Res}_{t=0} f(t)g(t)dt\right)\operatorname{Tr}(\alpha\beta)$$

$$\operatorname{Tr}: PV \to \mathbb{C} \qquad \mu \mapsto \int_X (\mu \lrcorner \Omega_X)\wedge\Omega_X$$

$\Longrightarrow (S, Q, \omega)$ dg symplectic space.

<u>Remk</u> $X = CY_3$ sub symplectic space

$$t\,PV^{0,0} \oplus PV^{1,1} \oplus t^{-1} PV^{2,2} \oplus t^{-2} PV^{3,3}$$

$$\updownarrow$$

$$\Omega^{3,0} \oplus \Omega^{2,1} \oplus \Omega^{1,2} \oplus \Omega^{0,3}$$

$\boxed{\text{Descendants Hodge wt}}$

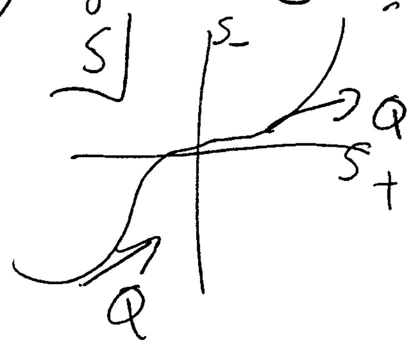
Lagrangian cone (after Givental)

$$\mathcal{J} : S_+ \longrightarrow S$$

$$\mu \longrightarrow t e^{\mu/t} - t$$

as a formal scheme (μ nilpotent)

① $\mathcal{J}(S_+)$ is a formal Lagrangian $\subset S$.



② Q is tangent to $\mathcal{J}(S_+)$

③ $\pi_+ : S \mapsto S_+$ $\quad \mathcal{J}(S_+) \xrightarrow{\pi} S_+$ formally diffeo.

④ $\mathcal{J}_*(Q, \{,\}) = Q$

pushforward nonlinear V.ectorfield, get linear one.

means $Q(te^{\mu/t}) = \left(Q\mu + \frac{1}{2}\{\mu,\mu\}\right)e^{\mu/t}$ (BTT Thm)

Defn: BCOV interaction I_X^{BCOV} is a local functional on $S_+ = PV(X)[[t]]$

$$\mathcal{P}(S_+) = \mathrm{Graph}\left(dI_X^{BCOV}\right)$$

(viewed for $S = T^*S_+$)

Lem [Costello-Li] $I_X(\mu) = \mathrm{Tr}\langle e^{\mu}\rangle_0$

$$\langle t^{k_1}\mu_1, \dots, t^{k_n}\mu_n\rangle_0 = \left(\int_{\overline{M}_{0,n}} \psi_1^{k_1} \cdots \psi_n^{k_n}\right) \mu_1 \wedge \cdots \wedge \mu_n$$

const map $(\mathbb{P}^1, X) \simeq \overline{M}_{0,n} \times X$

- Degenerate BV theory

$$S = \begin{matrix} S_+ \circlearrowleft Q \\ \oplus \\ S_- \circlearrowleft Q \end{matrix}$$

Failure of Q to preserve splitting at cochain level

$\Downarrow$

degenerate Poisson.

Let ω^{-1} be the Poisson kernel

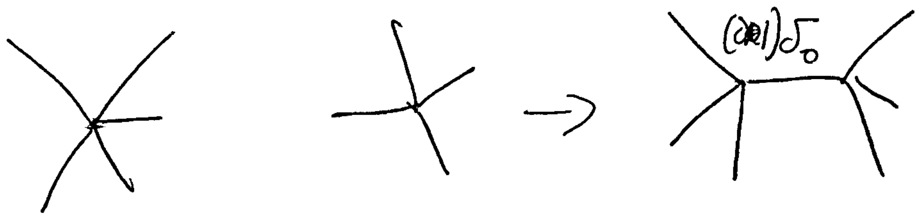
$$\omega^{-1} = \sum_k (-t)^k \otimes t^{-k-1} \delta_0$$

$$K_0 = \pi_+ \otimes \pi_+ \left((Q \otimes 1)\omega^{-1} \right) \quad \text{Poisson kernel.}$$
$$= (\partial \otimes 1)\delta_0 \qquad Q = \bar\partial + t\partial$$

degenerate because splitting not preserved.

<u>BV bracket</u>

$$\mathcal{O}_{loc}(S_+) \otimes \mathcal{O}_{loc}(S_+) \to \mathcal{O}_{loc}(S_+)$$

$\{\ ,\ \}$ 

<u>Lemma:</u> I_X^{BCOV} satisfies CME

$$Q I_X^{BCOV} + \frac{1}{2}\{I_X^{BCOV}, I_X^{BCOV}\} = 0$$

Geometrically

$$\pi_{+*}(Q) = Q + \{I_X^{BCOV}, -\}$$ defines L_∞ str in S_+

L_∞-quasi iso

$$S_+ \xrightarrow{\pi_+ \circ P} S_+$$

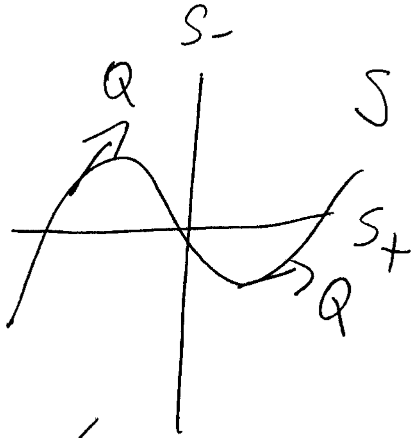
$$Q, \{,\} \longrightarrow Q + \{I_x, -\}$$

$$\mu \longrightarrow [te^{\mu/t} - t]_+$$

$\uparrow$ Not variational

$\uparrow$ is variational

$$\frac{Q, \{,\}}{S_+} \xrightarrow{P}$$



$\downarrow$ $\swarrow \pi_+$

$$\frac{Q + \{I_x, -\}}{S_+}$$

Quantization Toy Model (V, ω, Q)

$Q \curvearrowright V_+ \subset V$ linear lag ~~subspace~~ subcomplex

ω^{-1}: Poisson kernel.

Def^n: $\mathcal{W}(V) = \prod (V^*)^{\otimes n}[[\hbar]] / \sim$

$$[a,b] = a \otimes b - b \otimes a \sim \hbar\, \omega^{-1}(a,b) \qquad a, b \in V^*$$

$\mathrm{Ann}(V_+) \subset V^*$ the annihilator of V_+

$$= \{\varphi : V \to \mathbb{C} \mid \varphi(V_+) = 0\}$$

$$\Longrightarrow \mathrm{Fock}(V) := \mathcal{W}(V) / \mathcal{W}(V)\,\mathrm{Ann}(V_+)$$

$$\ni |0\rangle \longleftrightarrow 1$$

Q preserves V and ω^{-1} and V_+

$\Rightarrow (W(V), Q)$, $(\text{Fock}(V), Q)$

Quantization: a state ~~Fock~~
$|F\rangle \in \text{Fock}(V)$ s.t. $Q|F\rangle = 0$

Let $V = V_+ \oplus V_-$ isotropic splitting.

$$V_- \curvearrowleft Q$$

$$\Rightarrow V^* = V_+^* \oplus \text{Ann}(V_+)$$

$$\begin{tikzcd}
Q \curvearrowright \mathcal{O}(V_+) \simeq \widehat{\text{Sym}}(V_+^*) \arrow[r] \arrow[dr, "\Phi", "\cong"'] & W(V) \circlearrowleft Q \arrow[d] \\
& \text{Fock}(Q) \circlearrowleft Q
\end{tikzcd}$$

doesn't preserve Q

<u>Defn</u>: $I \in \mathcal{O}(V_+)[[\hbar]]$ satisfies QME
if $(Q + \hbar \partial_K) e^{I/\hbar} = 0$

$e^{I/\hbar} \Leftrightarrow |I\rangle = e^{I/\hbar}|0\rangle$

QME: $QI + \hbar\partial_K I + \frac{1}{2}\{I, I\} = 0 \Leftrightarrow Q|I\rangle = 0$

Quantum BCOV theory

$$(S = S_+ \oplus S_-, Q, \omega)$$

$$K_0 = (\partial \otimes 1)\delta$$

$\mathcal{O}(S_+)$: distributions

$\partial_{K_0} \curvearrowright \mathcal{O}(S_+)$ ill-defined.

homotopic RG flow $\Rightarrow$ $|I_r\rangle$ vary homotopically

$\Downarrow$

cohomology, doesn't move at.

$|I\rangle \in \mathrm{Fock}(H^*(S_+, Q))$

Do RG flow, $|I\rangle$ changes by Q-exact. $\leftarrow$ Can calculate this at any scale! ($r \to \infty$ is easier, maybe)

Get numbers: Choose a splitting $H^\cdot(S, Q) = H^*(S_+, Q) \oplus L$

$$\Longrightarrow \mathrm{Fock}(H^\cdot(S_+, Q)) \overset{L}{=} \mathcal{O}(H^\cdot(S_+, Q))$$

$$H^\cdot(S_+ = PV[[t]], Q) \overset{L}{\simeq} H^*(\wedge^\cdot T_X)[[t]]$$

Splitting on cochains is ok, but need L for cohomology

Ell curve: $E = \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau$

$$S_t = \Omega^{0,*}_E[[t]] \oplus \Omega^{0,*}_E(T_E[-1])[[t]]$$

$$b_k t^k \qquad \eta_k t^k$$

Stationary sector: $b_{>0} = 0 \quad \eta = \text{const}$

$$I_X = \sum_{k\geq 0} \int_E \left(\eta_k \frac{b_0^{k+2}}{(k+2)!}\right) \quad \text{sat. (CME)}$$

CME $\Leftrightarrow$ classical integrable hierarchy

QME $\Leftrightarrow$ Quantum integrable hierarchy

$$\Longrightarrow I_X^g = \sum_{k\geq 0} \int \eta_k \frac{W^{(k+2)}(b_0)}{k+2}$$

$$\left(W^{(k+2)}(b_0) = \sum \frac{k_i!}{\prod_i k_i!} \left(\prod_i \frac{1}{i!}\left(\sqrt{\hbar}\,\partial_z\right)^{i-1} b_0\right)^{k_i}\right.$$

scale zero (UV finite)

$\Longrightarrow$ generating function:

$$\mathrm{Tr}_{\text{Heisenberg vertex alg}}\, q^{L_0 - \frac{1}{24}} e^{\sum \eta_k \oint \frac{W^{(k+2)}}{k+2}}$$

$$q = \exp(2\pi i \tau)$$

mirror to $\langle t^{k_1}\omega_1, t^{k_2}\omega_2, \dots, t^{k_n}\omega_n \rangle$

$\mathrm{Tr}\ e^{-t\Delta}$
$S^1 \to X$

observe: $h_t(x,y) \sim \int D\gamma\, e^{-|\dot\gamma|^2}$

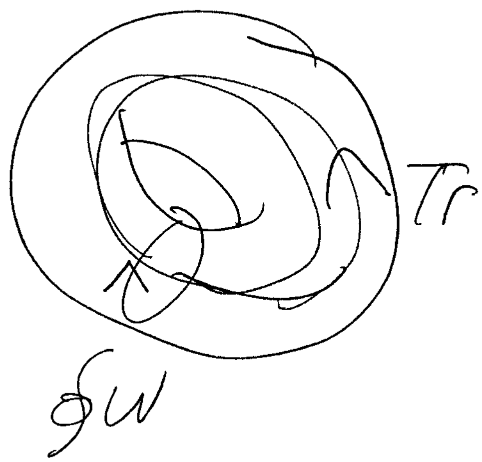
$\gamma : [0,t] \to X$

$\gamma(0) = x$

$\gamma(t) = y$

Trace $= \int_{\text{loopspace}} e^{-|\dot\gamma|^2}$

2d



$q^{L_0 - \frac{1}{24}}$

analogue of $e^{-t\Delta}$