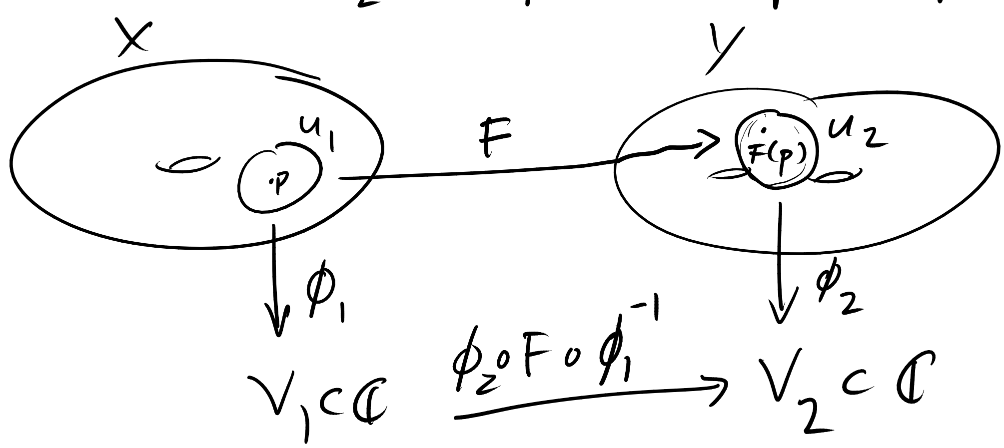
Holomorphic maps between Riemann surfaces.

Let X and Y be Riemann surfaces

<u>Def</u> A continuous map. $F: X \to Y$ is holomorphic a $p \in X$ iff there are charts $\phi_1 : U_1 \to V_1$ on X $p \in U_1$
$\phi_2 : U_2 \to V_2$ on Y $F(p) \in U_2$

such that $\phi_2 \circ F \circ \phi_1^{-1}$ is holomorphic at ϕ.



Put simiply both the source and target space look locally like $\mathbb{C}$, so it makes sense to say a map between them is holomorphic.

<u>Example</u>: $\{\text{holomorphic maps } X \to \mathbb{C}\} \cong \{\text{holomorphic functs on } X\}$

<u>Def</u> The category of Riemann surfaces $\mathcal{RS}$:
Objects : Riemann surfaces $X, Y, \dots$
Morphisms : $\mathrm{Mor}(X,Y) = \{\text{holomorphic maps } X \to Y\}$
Composition = composition of maps. (preserves holomorphicity)

Remark: as one goes deeper into algebraic geometry, the category-theoretic side becomes more prominent.

Let $\mathcal{O}_Y(W)$ denote set of holomorphic functions on open set $W \subset Y$
$\mathcal{O}_X(U)$ " " " " $U \subset X$
$\mathcal{M}_Y(W)$, $\mathcal{M}_X(U)$ meromorphic functions.

By precomposing with F: $F^* : \mathcal{O}_Y(W) \to \mathcal{O}_X(F^{-1}(W))$
$$g \longmapsto g \circ F$$

also $F^* : \mathcal{M}_Y(W) \to \mathcal{M}_X(F^{-1}(W))$
for every $W \subset Y$ open.

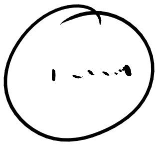
"holomorphic functions pull back along homorphic maps (morphisms)"
One could actually define holomorphic maps by this property.

Now we finally have the notion of isomorphism of Riemann surfaces.
$F : X \to Y$ is isomorphim if $\exists\, G : Y \to X$ such that
$GF = id_X$ and $FG = id_Y$.

<u>Proposition</u>: Let $F : X \to Y$ be a holomorphic map that is 1-1 and onto then F is an isomorphism. (Inverse is automatically holomorphic)

Facts from complex analysis:
<u>Open mapping theorem</u>: $F : X \to Y$ nonconstant holomorphic map.
then $U \subset X$ open $\Longrightarrow$ $F(U) \subset Y$ is open.

<u>Identity theorem</u>: $F, G : X \to Y$ X connected
If $F = G$ on a subset $S \subset X$ with a limit point in X
then $F = G$ on all of X.

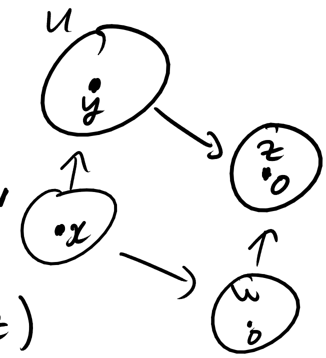


<u>Proposition</u> Let X be compact, Y connected. $F : X \to Y$ nonconstant
then Y is compact and F is onto.

<u>Proof</u> By open mapping thm $F(X)$ is open in Y. ← Y Hausdorff
But also X compact $\Rightarrow F(X)$ is compact. $\Longrightarrow F(X)$ is closed in Y.
Since Y is connected, an open and closed set is $\emptyset$ or Y. ■

<u>Proposition</u>: (Discreteness of preimages) $F: X \to Y$ nonconstant, Y connected.
For every $y \in Y$, $F^{-1}(y) \subset X$ is a discrete subset.
In particular, if X is compact, $F^{-1}(y)$ is finite and nonempty.

<u>Proof</u>: Fix local coordinate z centered at $y \in Y$
(so y corresponds to $z=0$)
For some $x \in F^{-1}(y)$. Choose local coord w centered at x.



Then F is expressed by the relation $w = g(z)$
where g is a holomorphic function and $0 = g(0)$.
Since the zeros of g are discrete, there is a neighborhood U' of $w=0$ such that g does not vanish except at $w=0$.
This neighborhood corresponds to a nbhd of $x \in X$ such that x is the only preimage of y in that neighborhood.

X compact + $F^{-1}(y)$ discrete $\to F^{-1}(y)$ compact $\Rightarrow F^{-1}(y)$ finite
Since Y is connected, we know Y is compact and F is onto.
so $F^{-1}(y)$ is never empty. ■

<u>Example</u>: there is a correspondence

$$\left\{ \begin{array}{l} \text{meromorphic functions} \\ \text{on } X \end{array} \right\} \cong \left\{ \begin{array}{l} \text{holomorphic maps} \\ F: X \to \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \\ \text{not constantly equal to } \infty \end{array} \right\}$$

$$\begin{array}{ccc} \Psi & & \\ f & \longmapsto & F(x) = \begin{cases} f(x), & x \text{ not a pole;} \\ \infty, & x \text{ a pole.} \end{cases} \end{array}$$

Reason: Near poles of f, one needs to check that $1/f$ is a holomorphic function, which it is if f is meromophic

Prop: $\mathbb{CP}^1$ is isomorphic to $\hat{\mathbb{C}}$.

$F : \mathbb{CP}^1 \to \hat{\mathbb{C}}$ is an isomorphism

$$F([z:w]) = \begin{cases} z/w, & w \neq 0 \\ \infty, & w = 0 \end{cases}$$

Inverse $F^{-1} : \hat{\mathbb{C}} \to \mathbb{CP}^1$ $\quad F^{-1}(\zeta) = \begin{cases} [\zeta : 1] & \zeta \neq \infty \\ [1:0] & \zeta = \infty \end{cases}$