Meromorphic functions on smooth plane curves.

$f \in \mathbb{C}[x,y]$ nonsingular irreducible polynomial
$X = \{(x,y) \in \mathbb{C}^2 \mid f(x,y) = 0\} \subset \mathbb{C}^2$

The x and y coordinates define functions
$$x, y : X \to \mathbb{C}$$
These functions are holomorphic

Hense any polynomial $g(x,y)$ defnis a holomorphic function on X as well.

This function may be zero: eg. the defining function $f(x,y)$ is zero at every point of f.

<u>Theorem</u> (Nullstellensatz) $f \in \mathbb{C}[x,y]$ irreducible
$X = V(f) = \{(x,y) \mid f(x,y) = 0\}$
let $h \in \mathbb{C}[x,y]$ then
h vanishes at every point of $X \iff f$ divides h.

<u>Corollary</u>: let $r(x,y) = \dfrac{g(x,y)}{h(x,y)}$ where g and h are polynomials, and f does not divide h. Then $r(x,y)$ defines a meromorphic function on X.

<u>Projective case</u>: let $F(x,y,z)$ be homogenous polynomial which is nonsingular and irreducible.

$$X = V(F) = \{[x:y:z] \in \mathbb{CP}^2 \mid F(x,y,z) = 0\} \subset \mathbb{CP}^2$$

let $G(x,y,z)$ be a homogeneous polynomial of degree m
$H(x,y,z)$ be " " of degree n

Then the ratio $R(x,y,z) = \dfrac{G(x,y,z)}{H(x,y,z)}$ has the property

$$R(\lambda x, \lambda y, \lambda z) = \frac{G(\lambda x, \lambda y, \lambda z)}{H(\lambda x, \lambda y, \lambda z)} = \frac{\lambda^m G(x,y,z)}{\lambda^n H(x,y,z)} = \lambda^{m-n} R(x,y,z)$$

So if $m=n$, $R(\lambda x, \lambda y, \lambda z) = R(x,y,z)$ and R actually defines a function on $\mathbb{CP}^2$ (at least at points where $H \neq 0$).

If H does not vanish identically on $X = V(F)$, then the restriction of $R(x,y,z)$ to X is a meromorphic function on X.

$F \nmid H$

Indeed, in chart where $z \neq 0$, we get affine coordinates by setting $z=1$

$$R(x,y,1) = \frac{G(x,y,1)}{H(x,y,1)}$$ and we are in the affine case.

Works similarly in $x \neq 0$ chart and $y \neq 0$ charts.

Smooth projective curves.

Here is a somewhat more intrinsic definition of a Riemann surface in projective space.

Let X be a Riemann surface, $X \hookrightarrow \mathbb{CP}^n$ a topological embedding.

$X \hookrightarrow \mathbb{CP}^n$ is a <u>holomorphic embedding</u> if

for every $p \in X$, ① there is a homogeneous coordinate such that $z_j \neq 0$, ② near p, $\left.\frac{z_k}{z_j}\right|_X$ a holomorphic function on X (with respect to the given complex structure on X). ③ there is a homogeneous coordinate z_i such that z_i/z_j is a local coordinate near p. (There is an open set $p \in U \subset X$ such that $z_i/z_j : U \to \mathbb{C}$ is a chart.)

A Riemann surface holomorphically embedding in $\mathbb{CP}^n$ is a <u>smooth projective curve</u>.

For a smooth projective curve $X \subset \mathbb{CP}^n$ $\frac{z_i}{z_j}$ is a meromorphic function on X, provided z_j is not identically zero on X.

Any rational combination of the functions $\frac{z_i}{z_j}$ for various i, j is also a meromorphic function on X.
Any such expression can be written as a ratio of two homogeneous polynomials $\frac{G(z_0, \ldots, z_n)}{H(z_0, \ldots, z_n)}$ of the same degree.

eg. $$\left(\frac{z_0}{z_1}\right)^2 + \frac{z_2}{z_3} = \frac{z_0^2 z_3 + z_2 z_1^2}{z_1^2 z_3}$$

<u>Thm</u>: If X is smooth projective curve in $\mathbb{CP}^n$, $G(z_0, \ldots, z_n)$ and $H(z_0, \ldots, z_n)$ homogeneous of degree d and G not identically zero on X, then $\frac{G}{H}$ is meromorphic on X.

The implicit function theorem shows that a curve
$X = V(F)$ (F nonsingular irreducible) is a smooth projective curve in the above sense.

Same is true if $F_1, \dots, F_{n-1}$ are a collection of homogeneous polynomials in $z_0, \dots, z_n$ defining $X = V(F_1, \dots, F_{n-1})$
the matrix $\left(\frac{\partial F_i}{\partial x_j}\right)_{\substack{i=1,\dots,n-1 \\ j=0,\dots,n}}$ has full rank at all $p \in X$.

<u>Examples</u> $\mathbb{CP}^1$ coordinates $[z:w]$

can map $\mathbb{CP}^1 \to \mathbb{CP}^3$

$$[z:w] \mapsto [z^3 : z^2 w : z w^2 : w^3]$$

Well-defined $[\lambda z : \lambda w] \mapsto [\lambda^3 z^3 : \lambda^3 z^2 w : \lambda^3 z w^2 : \lambda^3 w^3]$ ✓

This is a holomorphic embedding. $X = \text{image}(\mathbb{CP}^1)$ is a smooth projective curve.

* X is called a <u>twisted cubic curve</u>.

Can also map $\mathbb{CP}^1 \longrightarrow \mathbb{CP}^2$

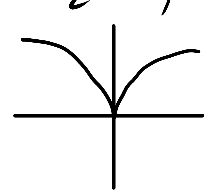
$$[z:w] \mapsto [z^3 : z^2 w : w^3]$$

This map is a homeomorphism onto its image (topological embedding) but it is not a holomorphic embedding. The image is defined by the homogeneous polynomial $F = z_0^2 z_2 - z_1^3$ which is singular:

$$\frac{\partial F}{\partial z_0} = 2 z_0 z_2 \qquad \frac{\partial F}{\partial z_1} = -3 z_1^2 \qquad \frac{\partial F}{\partial z_2} = z_0^2$$

all vanish at $[z_0 : z_1 : z_2] = [0:0:1]$

Affine picture: $z_2 \neq 0$, $x := \frac{z_0}{z_2}$, $y := \frac{z_1}{z_2}$ $\quad f = \frac{F}{z_2^3} = x^2 - y^3$

<u>Cusp</u>: $x^2 = y^3$:  (projective curve, not smooth)