More examples of meromorphic functions

Last time: Rational functions $h(z) = \frac{f(z)}{g(z)}$ $\quad f(z), g(z) \in \mathbb{C}[z]$
define meromorphic functions on
$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

Now: Proposition Every meromorphic function on $\hat{\mathbb{C}}$ is a rational function.

Proof let $f(z)$ be a meromorphic function on $\hat{\mathbb{C}}$

let $z_1, \dots, z_n \in \mathbb{C}$ be zeros of f $\quad \mathrm{ord}_{z_i}(f) = k_i > 0$

let $p_1, \dots, p_m \in \mathbb{C}$ be poles of f $\quad \mathrm{ord}_{p_i}(f) = -l_i < 0$

At all other $p \in \mathbb{C}$, $\mathrm{ord}_p(f) = 0$

$$\text{Let } h(z) = \prod_{i=1}^{n} (z - z_i)^{k_i} \Big/ \prod_{j=1}^{m} (z - p_j)^{l_j}$$

So $h(z)$ has same zeros/poles in $\mathbb{C}$

Thus $f(z)/h(z)$ is a meromorphic function and

$$\mathrm{ord}_{z_i}(f(z)/h(z)) = k_i - k_i = 0$$

$$\mathrm{ord}_{p_i}(f(z)/h(z)) = -l_i + l_i = 0$$

That is $\mathrm{ord}_p(f/h) = 0$ for all $p \in \mathbb{C}$.

Thus $g := f/h$ is an entire function with no zeros.

$$g(z) = \sum_{n=0}^{\infty} c_n z^n \quad \text{Valid in all of } \mathbb{C}.$$

Now g must also be meromorphic at ∞. The local coordinate is $w = 1/z$

so $g(z) = g(1/w) = \sum_{n=0}^{\infty} c_n w^{-n}$

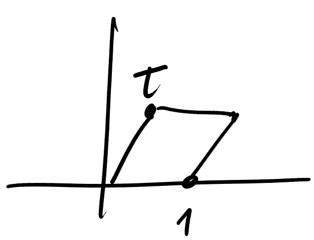
This series can only have finitely many terms
$\Rightarrow$ $g(z)$ is a polynomial
Since $g(z)$ has no zeros in $\mathbb{C}$ $\Rightarrow$ g is constant!

So $f = \underset{\text{const}}{g}\,\underset{\text{rational}}{h}$ $\blacksquare$

Corollary: f a meromorphic function on $\hat{\mathbb{C}}$, then
$$\sum_p \operatorname{ord}_p(f) = 0$$

Complex tori: $\mathbb{C}/L$, $L = \mathbb{Z} + \mathbb{Z}\tau$, $\operatorname{Im}\tau > 0$.

Since $\mathbb{C}/L$ is a quotient of $\mathbb{C}$, meromorphic functions on $\mathbb{C}/L$ are the same as L-invariant / L-periodic meromorphic functions on $\mathbb{C}$.



$f : \mathbb{C} \to \mathbb{C}$ is L-invariant/L-periodic if $\forall\, l \in L$, $f(z+l) = f(z)$

Any holomorphic L-periodic function is constant, but interesting meromorphic functions exist:

Ratios of Theta functions:

Fix τ, $\mathrm{Im}(\tau)>0$: The basic theta function is

$$\Theta(z)=\sum_{n=-\infty}^{\infty} e^{\pi i[n^2\tau+2nz]}$$

This function satisfies $\Theta(z+1)=\Theta(z)$
but $\Theta(z+\tau)\neq\Theta(z)$, instead, we find

$$\Theta(z+\tau)=e^{-\pi i[\tau+2z]}\Theta(z)$$

Nevertheless, $\Theta(z)=0 \Leftrightarrow \Theta(z+1)=0 \Leftrightarrow \Theta(z+\tau)=0$
so z_0 is a zero of $\Theta \Leftrightarrow z_0+m+n\tau$ is a zero.

<u>Fact</u>: the zeros of Θ are $\frac{1}{2}+\frac{\tau}{2}+m+n\tau$
and $\mathrm{ord}(\Theta)=1$ at these points.

Translated Θ function: $\Theta^{(x)}(z)=\Theta(z-\frac{1}{2}-\frac{\tau}{2}-x)$
Thus $\Theta^{(x)}(z)$ has zeros at $\{x+m+n\tau\}=x+L$.

Transforms as $\Theta^{(x)}(z+1)=\Theta^{(x)}(z)$

$$\Theta^{(x)}(z+\tau)=\underbrace{-e^{-2\pi i(z-x)}}\Theta^{(x)}(z).$$

This prefactor is the reason the function is not well-defined on $\mathbb{C}/L$.
To get rid of it, consider ratios.

Pick m points x_i n points y_j

and consider $R(z) = \dfrac{\prod_{i=1}^{n} \theta^{(x_i)}(z)}{\prod_{j=1}^{m} \theta^{(y_j)}(z)}$ Meromorphic on $\mathbb{C}$

Obvious that $R(z+1) = R(z)$. Need to check $R(z+\tau) = R(z)$

$$R(z+\tau) = \frac{\prod \theta^{(x_i)}(z+\tau)}{\prod \theta^{(y_i)}(z+\tau)} = (-1)^{m-n} \frac{\prod e^{-2\pi i(z-x_i)} \theta^{(x_i)}(z)}{\prod e^{-2\pi i(z-y_i)} \theta^{(y_i)}(z)}$$

$$= \underbrace{(-1)^{m-n} e^{-2\pi i\left[(m-n)z + \sum y_i - \sum x_i\right]}}_{\text{still have this prefactor.}} R(z)$$

Can get rid of it if we take $m=n$ and $\sum y_i - \sum x_i \in \mathbb{Z}$

Proposition: choose two sets of d complex numbers $\{x_i\}$ $\{y_j\}$ (repetitions allowed) such that

$$\sum_{i=1}^{d} x_i - \sum_{j=1}^{d} y_j \in \mathbb{Z}$$

Then $R(z) = \dfrac{\prod_{i=1}^{d} \theta^{(x_i)}(z)}{\prod_{j=1}^{d} \theta^{(y_j)}(z)}$ is an L-periodic meromorphic function on $\mathbb{C}$

Hence $R(z)$ defines a meromorphic function on $\mathbb{C}/L$.