# Holomorphic functions on Riemann surfaces.

$X$ a Riemann surface, $W \subset X$ open
$f : W \to \mathbb{C}$ a function

<u>Def</u> $f$ is holomorphic on $W$ if it is holomorphic when expressed in terms of a holomorphic coordinate.
That is, for any chart $\varphi : U \to V \subset \mathbb{C}$
the composition
$$f \circ \varphi^{-1} : \varphi(U \cap W) \longrightarrow \mathbb{C}$$
is holomorphic in the usual sense.

Because different charts are related by holomorphic transition functions, we can use any chart to check holomorphicity of $f : W \to \mathbb{C}$

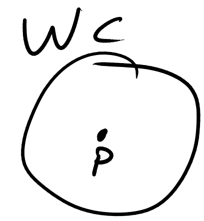
$$f \circ \varphi_1^{-1} = (f \circ \varphi_2^{-1}) \circ (\varphi_2 \circ \varphi_1^{-1})$$

<u>Example</u>: $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. $\infty \in W \subset \hat{\mathbb{C}}$.
$f : W \to \mathbb{C}$ is holomorphic at $\infty$
if $g(z) = \begin{cases} f(1/z) & z \neq \infty \\ f(\infty) & z = 0 \end{cases}$ is holomorphic at zero.

We can similarly use charts to define the notions of removable singularity, pole, and essential singularity.
$\to$ $f$ has the property iff $f \circ \varphi^{-1}$ has the property for each chart $\varphi$.

One can also give a coordinate-independent characterization of the three types of singularities.

$f : W \setminus \{p\} \to \mathbb{C}$ holomorphic




Singularity of $f$ at $p$ is:

Removable $\iff$ $\lim_{x\to p} |f(x)|$ exists and is finite

pole $\iff$ $\lim_{x\to p} |f(x)| = \infty$

essential $\iff$ $\lim_{x \to p} |f(x)|$ does not exist.

A function $f : W \to \mathbb{C}$ is <u>meromorphic</u> if at each point $p \in W$ it is either holomorphic, has a removable singularity, or has a pole.

<u>Laurent series:</u> Let $f : W \setminus \{p\} \to \mathbb{C}$ be a function with a singularity at $p$. (and otherwise holomorphic)
Let $\varphi : U \to V$ be a chart so that $p \in U \subset W$ and $\varphi(p) = z_0$
Then $f(\varphi^{-1}(z))$ has a Laurent series

$$f(\varphi^{-1}(z)) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

The coefficients $c_n$ depend on the choice of local coordinates
But we can define:

$$\mathrm{ord}_p(f) = \inf \{ n \mid c_n \neq 0 \}$$

($p$ is essential singularity $\iff$ $\mathrm{ord}_p(f) = -\infty$ )
$p$ removable singularity $\iff$ $\mathrm{ord}_p(f) \geq 0$
$p$ pole $\iff$ $\mathrm{ord}_p(f) < 0$ and finite

We should check that the order of a function does not depend on the choice of chart.

If $z=\varphi_1(x)$ and $w=\varphi_2(x)$ are two coordinates,
$z_0=\varphi_1(p)$ $\quad w_0=\varphi_2(p)$

The transition function $T=\varphi_1\circ\varphi_2^{-1}$
Has the form

$$z=T(w)=z_0+\sum_{n=1}^{\infty}a_n(w-w_0)^n$$

Suppose $(f\circ\varphi_1^{-1})(z) = c_{n_0}(z-z_0)^{n_0}+\sum_{n>n_0}c_n(z-z_0)^n$

$$(f\circ\varphi_2^{-1})(w) = f\circ\varphi_1^{-1}\circ\varphi_1\circ\varphi_2^{-1}(w) = (f\circ\varphi_1^{-1})(T(w))$$

$$= c_n\Big(\sum_{m=1}^{\infty}a_m(w-w_0)^m\Big)^{n_0}+\sum_{n>n_0}c_n\Big(\sum_{m=1}^{\infty}a_m(w-w_0)^m\Big)^n$$

$$= c_n a_1^{n_0}(w-w_0)^{n_0} + \text{higher order terms.}$$

Since $T$ is a holomorphic homeomorphism $a_1\neq 0$
Thus $c_n a_1^{n_0}\neq 0$ and the order of $f\circ\varphi_2^{-1}$ is also $n_0$

Key properties: let $f, g$ be meromorphic functions

$$\operatorname{ord}_p(fg)=\operatorname{ord}_p(f)+\operatorname{ord}_p(g)$$
$$\operatorname{ord}_p(1/f) = -\operatorname{ord}_p(f)$$
$$\operatorname{ord}_p(cf)=\operatorname{ord}_p(f)\ ,\quad c\in\mathbb{C}\setminus\{0\}$$

$$\operatorname{ord}_p(f+g)\geq \min\big(\operatorname{ord}_p(f),\operatorname{ord}_p(g)\big)$$
and $\operatorname{ord}_p(f+g)=\min\big(\operatorname{ord}_p(f),\operatorname{ord}_p(g)\big)$
provided $\operatorname{ord}_p(f)\neq\operatorname{ord}_p(g)$.

Example: A rational function $h(z) = \frac{f(z)}{g(z)}$ $\quad f(z), g(z) \in \mathbb{C}[z]$
defines a meromorphic function on the Riemann sphere. $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

$$h(z) = c \, \frac{(z-a_1)^{k_1} \cdots (z-a_n)^{k_n}}{(z-b_1)^{\ell_1} \cdots (z-b_m)^{\ell_m}} \qquad a_i \neq b_j$$

Finite zeros: $a_1, \ldots, a_n$ ; Finite poles: $b_1, \ldots, b_m$
$\mathrm{ord}_{a_i}(h) = k_i \qquad \mathrm{ord}_{b_j}(h) = -\ell_j$

What about $\infty \in \hat{\mathbb{C}}$? $\quad \mathrm{ord}_\infty(h) = \deg(g) - \deg(f) = \sum \ell_j - \sum k_i$

At all other points $\mathrm{ord}_p(h) = 0$.

More properties

- zeros and poles of a meromorphic function are discrete sets
- If $f$ is a meromorphic function on a <u>compact</u> Riemann surface, then the zeros and poles of $f$ are <u>finite</u> sets.

- Maximum modulus principle: $W$ connected open subset of Riemann surface $X$. $f: W \to \mathbb{C}$ holomorphic. If $\exists\, p \in W$ such that $|f(x)| \le |f(p)|$ $\forall x \in W$, then $f$ is constant in $W$.
- If $X$ is <u>compact</u> and $f: X \to \mathbb{C}$ is holomorphic on all of $X$, then $f$ is constant.
  (indeed $|f(x)|$ is continuous, and must achieve its maximum at some $p \in X$ since $X$ is compact)