lecture 5

Projective spaces: $\mathbb{CP}^n$ = set of 1-dim. subspaces of $\mathbb{C}^{n+1}$

$\vec{z} = (z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1}$ If $\vec{z} \neq 0$ then $[\vec{z}] = \mathrm{span}_{\mathbb{C}}\{\vec{z}\} \in \mathbb{CP}^n$

$[\vec{z}] = [z_0 : z_1 : \cdots : z_n]$

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$$
$$\vec{z} \mapsto [\vec{z}]$$
$$[\vec{z}] = [\vec{z}\,'] \iff \vec{z}\,' = \lambda \vec{z} \quad \text{for } \lambda \in \mathbb{C}^*$$
$$\mathbb{CP}^n = \left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \mathbb{C}^* \quad \text{(use quotient topology)}$$

$\mathbb{CP}^n$ is a complex manifold of complex dimension n.

Atlas: $U_i = \{ [z_0 : \cdots : z_n] \mid z_i \neq 0 \}$ $\quad i = 0, 1, \dots, n$

$$\varphi_i : U_i \to \mathbb{C}^n \qquad [z_0 : z_1 \cdots z_n] \mapsto \left(\frac{z_0}{z_i}, \dots \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i} \dots, \frac{z_n}{z_i}\right)$$

$$U_i \cap U_j = \{ z_i \neq 0, z_j \neq 0 \}$$
$$\varphi_j \circ \varphi_i^{-1} = \text{multiplication by } \frac{z_i}{z_j}$$

- $z_0, \dots, z_n$ are called homogeneous coordinates. z_i is not a well defined function on $\mathbb{CP}^n$, but ratios z_i / z_j are well-defined.

Homogeneous polynomials A polynomial $F(z_0, z_1, \dots, z_n)$ is called homogeneous of degree d if every term has total degree d.

e.g. $\quad z_0 z_1^2 + z_0 z_1 z_2 + z_2^3$ - homogeneous of degree 3.

$$F(\lambda z_0, \dots, \lambda z_n) = \lambda^d F(z_0, \dots, z_n)$$

The value of F at a point in $\mathbb{CP}^n$ does not make sense, but the condition $F(z_0, \dots, z_n) = 0$ does make sense.

hypersurface: $V(F) = \{[z_0 : \dots : z_n] \in \mathbb{CP}^n \mid F(z_0, \dots, z_n) = 0\}$

complete intersection: $V(F_1, \dots, F_r) = V(F_1) \cap \dots \cap V(F_r)$

Such sets are examples of <u>projective varieties</u>.

<u>Affine charts and (de)homogenization.</u>

$$U_0 = \{[z_0 : \dots : z_n] \in \mathbb{CP}^n \mid z_0 \neq 0\} = \mathbb{CP}^n \setminus V(z_0)$$

Affine coordinates $Z_i := \frac{z_i}{z_0}$ $\qquad \varphi_0 : U_0 \to \mathbb{C}^n$

Homogeneous $\qquad\qquad$ Non homogeneous

$$F(z_0, \dots, z_n) \longmapsto f(Z_1, \dots, Z_n) := \frac{F(z_0, \dots, z_n)}{z_0^d} = F(1, Z_1, \dots, Z_n)$$

$$G(z_0, \dots, z_n) := z_0^d \, g\left(\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right) \longleftarrow g(Z_1, \dots, Z_n)$$

$$z_0^3 + z_1^3 + z_2^3 \longrightarrow 1 + Z_1^3 + Z_2^3$$

$$z_1^2 + z_2 z_0 + 4 z_0^2 \longleftarrow Z_1^2 + Z_2 + 4$$

If $X = V(F) \subset \mathbb{CP}^n$ then $\varphi_0(X \cap U_0) = V(f) \subset \mathbb{C}^n$

dehomogenization of F.

Projective plane curves

$n=2$ $\quad \mathbb{CP}^2$ homogeneous coordinates $[z_0 : z_1 : z_2]$
$= [x : y : z]$

Def Let $F(x,y,z)$ be homogeneous polynomial.
F is nonsingular if
$F, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ do not all vanish simultaneously
at any point of $\mathbb{CP}^2$.
(Remark: $F(\vec{0}) = \frac{\partial F}{\partial x}(\vec{0}) = \frac{\partial F}{\partial y}(\vec{0}) = \frac{\partial F}{\partial z}(\vec{0}) = 0$
but $\vec{0}$ doesn't represent a point of $\mathbb{CP}^2$]

Thm If $F(x,y,z)$ is nonsingular then $X = V(F)$ is a Riemann surface.

Sketch proof: F nonsingular $\Rightarrow$ dehomogenizations are nonsingular.

$$X = \underbrace{(X \cap U_0)}_{\uparrow} \cup \underbrace{(X \cap U_1)}_{\uparrow} \cup \underbrace{(X \cap U_2)}_{\uparrow}$$

these are Riemann surfaces.

Note: F homogeneous of degree $d \Rightarrow F = \frac{1}{d} \sum_{i=0}^{n} z_i \frac{\partial F}{\partial z_i}$

so $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} = dF$

so $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = F = 0 \Rightarrow \frac{\partial F}{\partial z} = 0.$

Higher dimensional projective curves.

$$X = V(F_1, \dots, F_{n-1}) \subset \mathbb{CP}^n$$
$$\begin{array}{ccc} \uparrow & & \uparrow \\ d_1 & \cdots & d_{n-1} \end{array}$$

<u>Complete intersection</u> of type $(d_1, \dots, d_{n-1})$

<u>Nonsingularity condition</u>: Matrix $\left(\dfrac{\partial F_i}{\partial x_j}\right)_{\substack{i=1,\dots,n-1 \\ j=0,\dots,n}}$

has maximal rank $= n-1$ at every point of X.

More generally, a <u>local complete intersection</u> is a curve that is locally of this form.

$\mathbb{CP}^n$ is compact since the map $\pi : S^{2n+1} \to \mathbb{CP}^n$, $S^{2n+1} = \{\vec{z} \in \mathbb{C}^{n+1} \mid \|\vec{z}\| = 1\}$, presents it as the continuous image of a compact space.

$V(F_1, \dots, F_r) \subset \mathbb{CP}^n$ is closed. so compact.

So nonsingular $V(F) \subset \mathbb{CP}^2$ or $V(F_1, \dots, F_{n-1}) \subset \mathbb{CP}^n$ are <u>compact</u> Riemann surfaces.

If $n=2$, $F(x,y,z)$ irreducible $\Rightarrow V(F) \subset \mathbb{CP}^2$ connected

<u>Ex</u> Plane cubic $y^2 = x^3 - x$ $\quad f(x,y) = y^2 - x^3 + x$

homogeneous $F(x,y,z) = y^2 z - x^3 + xz^2$

Points of $V(y^2 z - x^3 + xz^2) \subset \mathbb{CP}^2$

Affine chart $U = \{z \neq 0\} \cong \mathbb{C}^2$ $\quad \mathbb{CP}^2 \setminus U \overset{V(z)}{=}$ "line at infinity"

A point $[x:y:z] \in U$ can be normalized to $[x:y:1]$

$$0 = F(x,y,1) = y^2 - x^3 + x$$

$$V(F) \cap U = \{ [x:y:1] \mid y^2 = x^3 - x \}$$

Rest of the points occur when $z = 0$

$$V(F) \cap V(z) = \left\{ [x:y:z] \;\middle|\; \begin{array}{c} z=0 \\ y^2 z = x^3 - xz^2 \end{array} \right\}$$

$$= \{ [x:y:0] \mid 0 = x^3 \}$$

$$= \{ [0:y:0] \} = \{ [0:1:0] \}$$

So $V(F) \cap V(z)$ contains just one point.

"The projective curve has one point at infinity".

<u>Eg.</u> Affine eqn $\quad x^2 + y^2 = 1$

Homog. eqn $\quad x^2 + y^2 = z^2$

Points at ∞ $\quad \{ [x:y:0] \mid x^2 + y^2 = 0 \}$

solutions are $y = \pm i x$

This only amounts to two points in $\mathbb{CP}^2$

$[1 : i : 0]$ and $[1 : -i : 0]$