Abel-Jacobi Maps and a geometric interpretation of Riemann-Roch.

Recall: $Cl(X) = Div(X)/PDiv(X)$
$Cl_0(X) = Div_0(X)/PDiv(X)$

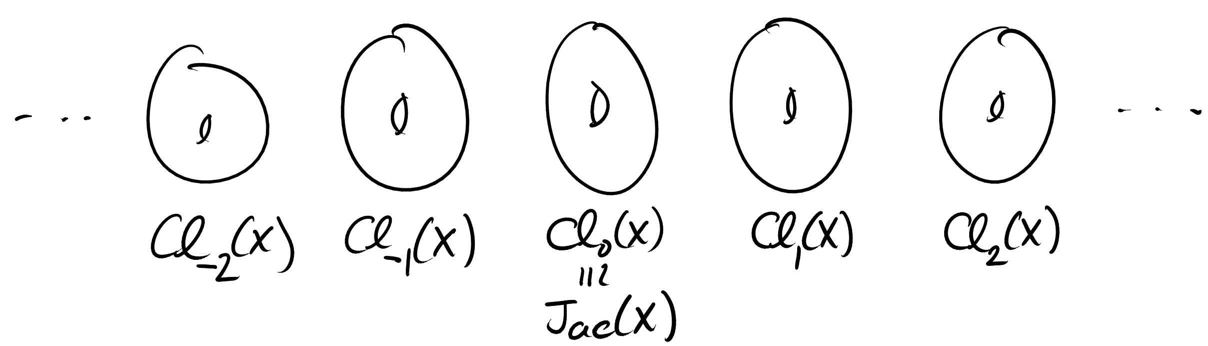
Exact seq: $0 \to Cl_0(X) \to Cl(X) \xrightarrow{deg} \mathbb{Z} \to 0$

Jacobian: $Jac(X) = \Omega^1(X)^*/\pi(H_1(X)) \cong \mathbb{C}^g/\Lambda$

Abel-Jacobi map: $A_0 : Div_0(X) \longrightarrow Jac(X)$
induces isomorphism
$A_0 : Cl_0(X) \xrightarrow{\cong} Jac(X)$

Now $Cl(X) = \coprod_{d \in \mathbb{Z}} Cl_d(X)$, where $Cl_d(X)$ is degree d

divisors modulo linear equalence.
$Cl_d(X)$ is a coset $[D] + Cl_0(X)$, where D is any divisor of degree D. Thus $Cl_d(X)$ is in bijection with $Cl_0(X)$, hence with $Jac(X)$. We can visualize $Cl(X)$ as a disjoint union of g-dimensional complex tori.



We can get maps $A_d : Div_d(X) \longrightarrow Jac(X)$
if we make some choices.

Choose a basepoint $p_0 \in X$. Define a map.

$$A_{p_0,d} : \mathrm{Div}_d(X) \longrightarrow \mathrm{Jac}(X)$$

$$A_{p_0,d}(D) = A_0(D - dp_0) \quad \text{for degree } d \text{ divisor } D.$$

This descends to a map $A_{d,p_0} : Cl_d(X) \to \mathrm{Jac}(X)$ which is a <u>bijection</u>.

Back to linear systems: $D \in \mathrm{Div}(X)$.

$$|D| = \{ E \in \mathrm{Div}(X) \mid E \geq 0,\ E \sim D \}$$

and this set has the structure of a projective space.

If $\deg(D) = d$ then $|D| \subseteq \mathrm{Div}_d(X)$

and the Abel-Jacobi image of $|D|$ is a single point.

$$\begin{array}{lccccc} A_{d,p_0} : & \mathrm{Div}_d(X) & \longrightarrow & Cl_d(X) & \xrightarrow{\cong} & \mathrm{Jac}(X) \\ & |D| & \longrightarrow & [D] & \longrightarrow & A_0(D - dp_0) \end{array}$$

[Two $E_1, E_2 \in |D|$ differ by a principal divisor, and A_0 sends principal divisors to zero]

<u>Symmetric products</u>: A divisor E that satisfies $E \geq 0$ can be though of as an unordered set of points in X, with multiplicities. Thus we can think of E as an element of the symmetric product of X:

$$\mathrm{Sym}^d X := X^{(d)} = \underbrace{X \times \cdots \times X}_{d} / S_d$$

where the symmetric group S_d permutes the factors

There is an embedding $\mathrm{Sym}^d X \hookrightarrow \mathrm{Div}_d(X)$
whose image is the set of divisors E st. $E \geq 0$.

Also if $D \in \mathrm{Div}_d(X)$, $|D| \subseteq \mathrm{Sym}^d X$

And in fact $\mathrm{Sym}^d X = \bigcup_{D \in \mathrm{Div}_d(X)} |D|$

<u>Fact</u> When X is a Riemann surface, $\mathrm{Sym}^d X$ is a complex manifold.

There is a map $AJ : \mathrm{Sym}^d X \longrightarrow \mathrm{Jac}(X)$

$$\begin{tikzcd} \mathrm{Sym}^d X \arrow[r, hook] \arrow[rr, bend right, "AJ"'] & \mathrm{Div}_d(X) \arrow[r, "A_{p_0,d}"] & \mathrm{Jac}(X) \end{tikzcd}$$

Fact: This is a <u>holomorphic map.</u>

The fibers of AJ are projective spaces, namely the complete linear systems of degree d.

$$AJ^{-1}(A_{p_0,d}(D)) = |D|$$

Degree 1: This is a map $AJ : X \to \mathrm{Jac}(X)$

$$AJ(p) = A_0(p - p_0)$$

<u>Prop</u>: If $g \geq 1$, $AJ : X \to \mathrm{Jac}(X)$ is injective.

<u>Proof</u> Suppose $AJ(p) = AJ(q)$. Then $A_0(p-p_0) = A_0(q-q_0)$ so $A_0(p-q) = 0$ so $p-q \in PDiv(X)$ by Abel's theorem

But we know $\ell(p) = 1$ if $g \geq 1$. Thus $|p|$ is a zero-dim'l projective space, i.e. just a point. $|p| = \{p\}$ So if $p - q \in PDiv(X)$, $q \in |p|$ and so $q = p$. ☒

<u>Cor</u>: If $g = 1$, $AJ : X \to \mathrm{Jac}(X)$ is an isomorphism.

<u>Proof</u> $\dim \mathrm{Jac}(X) = g = 1$, so $\mathrm{Jac}(X)$ is a Riemann surface. AJ is an injective holomorphic map between R.S.'s, so it is an isomorphism.

Thus, every genus one Riemann surface is a complex torus!

<u>Relation to Riemann-Roch</u>:

$AJ : \mathrm{Sym}^d X \to \mathrm{Jac}(X)$ is a holomorphic map Between complex manifolds of dimensions d and g respectively Therefore, the dimension of its fibers is bounded below by the difference in dimensions.

$$\boxed{\dim(\text{fiber of } AJ) \geq d - g}$$

But the fibers of AJ are the complete linear systems of degree d!
Thus, we have shown that if D is a divisor of degree d,

$$\dim |D| \geq d - g = \deg D - g$$

or

$$\ell(D) = \dim L(D) = \dim |D| + 1 \geq \deg D + 1 - g$$

Thus we have derived Riemann's inequality.

$$\boxed{\ell(D) \geq \deg D + 1 - g}$$

Roch's contribution was to figure out the error term

$$\ell(D) = \deg D + 1 - g + \ell(K - D)$$