Jacobians and Abel's theorem 1

X - compact connected Riemann surface.

$$\mathrm{Div}(X) \supseteq \mathrm{Div}_0(X) \supseteq \mathrm{PDiv}(X)$$

Divisor class group: $\mathrm{Cl}(X) = \mathrm{Div}(X)/\mathrm{PDiv}(X)$ [Aka. $\mathrm{Pic}(X)$
$\mathrm{Cl}_0(X) = \mathrm{Div}_0(X)/\mathrm{PDiv}(X)$ Picard group]

There is an exact sequence

$$0 \longrightarrow \mathrm{Div}_0(X) \longrightarrow \mathrm{Div}(X) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0$$

and

$$0 \rightarrow \mathrm{Cl}_0(X) \rightarrow \mathrm{Cl}(X) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0$$

This splits, so $\mathrm{Cl}(X) \cong \mathrm{Cl}_0(X) \times \mathbb{Z}$ (noncanonically)

To understand $\mathrm{Cl}(X)$, the main thing is to get at $\mathrm{Cl}_0(X)$.

<u>Abel's theorem</u> (weak/simplified version)

Suppose X has genus g. Then there is an isomorphism of abelian groups

$$\boxed{\mathrm{Cl}_0(X) \cong \mathbb{R}^{2g}/\mathbb{Z}^{2g}}$$

Eg. $g=0$ $\mathrm{Cl}_0(X) = \{0\}$ is a trivial group
$g=1$ $\mathrm{Cl}_0(X) = \mathbb{R}^2/\mathbb{Z}^2$ is a torus.

In fact, the $2g$-dim'l torus $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$ carries the structure of a complex manifold making it a complex torus $\mathbb{C}^g/\Lambda$ or <u>abelian variety</u>. This complex structure <u>depends on</u> the complex structure of X.

Recall: $C_0(X)$ 0-chains $\quad (\partial : C_0(X) \to 0)$
$C_1(X)$ 1-chains $\quad \partial : C_1(X) \to C_0(X)$
$C_2(X)$ 2-chains $\quad \partial : C_2(X) \to C_1(X)$

$Z_i(X) = \ker \partial$
$B_i(X) = \operatorname{Im} \partial$

$$H_0(X) = \frac{Z_0(X)}{B_0(X)} = \frac{C_0(X)}{B_0(X)} \cong \mathbb{Z}$$

$$H_1(X) = \frac{Z_1(X)}{B_1(X)} \cong \mathbb{Z}^{2g}$$

The isomorphism $H_0(X) \cong \mathbb{Z}$ is given by degree

$\boxed{\text{A zero chain (=divisor) is a boundary iff it has degree zero}}$

If $\omega \in \Omega^1(X)$ is a holomorphic 1-form then $d\omega = 0$
Thus, by Stokes, For any 1-chain α and 2-chain β,

$$\int_{\alpha + \partial\beta} \omega = \int_\alpha \omega + \int_{\partial\beta} \omega = \int_\alpha \omega + \int_\beta d\omega = \int_\alpha \omega + 0$$

Thus homologous 1 chains have the same value of $\int_\alpha \omega$.

so we can think of integration as defining a bilinear map

$$\int : \frac{C_1(X)}{B_1(X)} \times \Omega^1(X) \longrightarrow \mathbb{C}$$

Restricting to 1-cycles, we get a bilinear map

$$\int : H_1(X) \times \Omega^1(X) \longrightarrow \mathbb{C}$$

An expression of the form $\int_a \omega$ with $a \in Z_1(X)$ or $H_1(X)$ and $\omega \in \Omega^1(X)$ is called a <u>period</u>.

For each $a \in H_1(X)$, taking the period on a gives a map

$$\Pi_a : \Omega^1(X) \to \mathbb{C} \qquad \text{or} \quad \Pi_a \in \Omega^1(X)^*$$
$$\omega \mapsto \int_a \omega$$

This amounts to a linear map $\Pi : H_1(X) \to \Omega^1(X)^*$
$$a \mapsto \Pi_a$$

<u>Def</u> The Jacobian of X is the abelian group

$$\boxed{\mathrm{Jac}(X) := \Omega^1(X)^* / \Pi(H_1(X))}$$

In more concrete terms, let $\omega_1, \ldots, \omega_g$ be a basis of $\Omega^1(X) \cong \mathbb{C}^g$
let $c_1, c_2, \ldots, c_{2g}$ be a basis of $H_1(X) \cong \mathbb{Z}^{2g}$.

Form the $g \times (2g)$ matrix
$$\Pi = \begin{pmatrix} \int_{c_1}\omega_1 & \int_{c_2}\omega_1 & \cdots & \int_{c_{2g}}\omega_1 \\ \int_{c_1}\omega_2 & & \ddots & \vdots \\ \vdots & & & \\ \int_{c_1}\omega_g & \cdots & & \int_{c_{2g}}\omega_g \end{pmatrix}$$

$$\left\{ \begin{aligned} &\Pi_{ij} = \int_{c_j} \omega_i \\ &1 \le i \le g \\ &1 \le j \le 2g \end{aligned} \right\}$$

Then Π is called the <u>period matrix of X</u> (w/r/t the chosen bases.)

let $\Lambda \subseteq \mathbb{C}^g$ denote the subgroup generated by the columns of Π. It is a fact that the columns are linearly independent over $\mathbb{R}$, so this is a lattice.

Prop: $\mathrm{Jac}(X) \cong \mathbb{C}^g/\Lambda$ is a complex torus of complex dim g.

Abel-Jacobi map: This is a map $A_0 : \mathrm{Div}_0(X) \to \mathrm{Jac}(X)$

Start with $D \in \mathrm{Div}_0(X)$. Since $\mathrm{Div}_0(X) = B_0(X)$, the group of boundary 0-chains, there exists a nonunique 1-chain $\gamma \in C_1(X)$ such that $\partial\gamma = D$

Two choices γ_1 and γ_2 differ by $\delta = \gamma_2 - \gamma_1$ that satisfies

$$\partial\delta = \partial\gamma_2 - \partial\gamma_1 = D - D = 0$$

ie $\delta \in Z_1(X)$.

Define $A_0(D) = \left[\int_\gamma\right] \in \mathrm{Jac}(X)$

That is, $\int_\gamma : \Omega^1(X) \to \mathbb{C}$ or $\int_\gamma \in \Omega^1(X)^*$, and we take the equivalence class $\left[\int_\gamma\right] \in \Omega^1(X)^*/\Pi(H_1(X)) = \mathrm{Jac}(X)$

We need to see that this does not depend on the choice of γ s.t. $\partial\gamma = D$. But if $\gamma_2 = \gamma_1 + \delta$ with $\delta \in Z_1(X)$

Then $\int_{\gamma+\delta} = \int_\gamma + \int_\delta = \int_\gamma + \Pi_\delta$

and $\Pi_\delta \in \Pi(H_1(X))$ so the equivalence class is the same.

Abel's theorem: $A_0 : \mathrm{Div}_0(X) \longrightarrow \mathrm{Jac}(X)$ is surjective, and its kernel is precisely $\mathrm{PDiv}(X)$.

$$0 \to \mathrm{PDiv}(X) \longrightarrow \mathrm{Div}_0(X) \xrightarrow{A_0} \mathrm{Jac}(X) \to 0$$

So A_0 induces an isomorphism $\mathrm{Cl}_0(X) \xrightarrow{A_0} \mathrm{Jac}(X)$.