Finding Equations for Projective Curves

Suppose D is a divisor on X that defines an embedding
$\phi_D : X \longrightarrow \mathbb{P}^n$

$R_n = \mathbb{C}[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n$

Set $R_{n,k} = \mathbb{C}[x_0, \ldots, x_n]_k$ be the subspace of degree k polynomials.

Let $\mathcal{I}_k = \mathcal{I}_k(X, \phi_D) \subseteq P(n,k)$ be the set of homogeneous polynomials that vanish on the image $\phi_D(X)$

$$R_n = \bigoplus_{k=0}^{\infty} R_{n,k} \quad \text{is a graded ring} \quad [R_{n,k} \cdot R_{n,\ell} \subset R_{n,k+\ell}]$$

$$\mathcal{I} = \bigoplus_{k=0}^{\infty} \mathcal{I}_k \quad \text{is a homogeneous ideal.} \quad [R_{n,k} \cdot \mathcal{I}_\ell \subset \mathcal{I}_{k+\ell}]$$

We want to show that $\mathcal{I}$ is nonzero, and to find some elements $F \subset \mathcal{I}_k$. Then F is a homogeneous poly that vanishes on $\phi_D(X)$, and $\phi_D(X) \subseteq V(F)$.

Restriction map: Pick a coordinate x_i that does not vanish identically on $\phi_D(X)$.

Note that $\operatorname{div}(x_i) \sim D$. We might as well assume $D = \operatorname{div}(x_i)$. as this doesn't change $\phi_D : X \to \mathbb{P}^n$ in an essential way.

Define a map
$$R_{n,k} \longrightarrow L(kD)$$
$$F \longmapsto \left(\frac{F}{x_i^k}\right) \circ \phi$$

Now $\operatorname{div}\left(\frac{F}{x_i^k}\right) = \operatorname{div}(F) - \operatorname{div}(x_i^k) = \operatorname{div}(F) - kD$

so $\operatorname{div}\left(\frac{F}{x_i^k}\right) + kD = \operatorname{div}(F) > 0$ So the map has target $L(kD)$ as claimed

In fact, there is a graded ring structure on

$$R(X,D) = \bigoplus_{k=0}^{\infty} L(kD)$$

The maps $R_{n,k} \to L(kD)$ have kernel $\mathcal{I}_k$:

$$0 \to \mathcal{I}_k(X, \phi_D) \to R_{n,k} \to L(kD) \to 0$$

And assemble into a graded ring map $R_n \to R(X,D)$ whose kernel is the ideal $\mathcal{I}$.

$$0 \to \mathcal{I}(X, \phi_D) \to R_n \longrightarrow R(X,D) \to 0$$

<u>Dimension counting</u>: <u>lemma</u> $\dim_{\mathbb{C}} R_{n,k} = \binom{n+k}{n} = \binom{n+k}{k}$

<u>Proof</u> consider an alphabet with two letters x, |. Words on this alphabet are expressions like

xxx|xx|x||xxxx

Such a word corresponds to a monomial in $\mathbb{C}[x_0, \ldots, x_n, \ldots]$

$$\text{xxx|xx|x||xxxx} \to x_0x_0x_0|x_1x_1|x_2||x_4x_4x_4x_4$$

$$= x_0^3 x_1^2 x_2 x_4^4 \in \mathbb{C}[x_0, x_1, x_2, x_3, x_4]$$

The bar | tells us to increment the index.

The space $\mathbb{C}[x_0, x_1, \dots, x_n]$ corresponds to words with <u>exactly</u> n bars $|$.

The subspace $R_{n,k}$ corresponds to words with <u>exactly</u> k x's.

So $\dim R_{n,k}$ = words with n bars and k x's $= \binom{n+k}{n} = \binom{n+k}{k}$ $\boxtimes$

On the other hand,

$$\dim L(kD) = k \cdot \deg(D) + 1 - g + \ell(K - kD)$$

$$= \deg(D) \cdot k + 1 - g \quad \text{if } k \text{ suff large.}$$

But $\binom{n+k}{k} = \dfrac{(n+k)!}{n!\,k!} = \dfrac{(n+k)\cdots(1+k)}{n!} = \dfrac{k^n}{n!} + \text{lower order terms.}$

As $k \to \infty$, $\dfrac{k^n}{n!}$ grows faster than $\deg(D) \cdot k$ (if $n > 1$)

And $\dim \ell_k \geq \binom{n+k}{k} - \ell(kD) > 0$ for k large enough.

So there are definitely homogeneous polys that vanish on $\phi_D(X)$!

$X: g = 1,\ D = 3p \qquad \ell(kD) - \ell(K - kD) = \deg(kD) + 1 - g$

$$\ell(kD) = 3k \quad (k \geq 1)$$

$\phi_D : X \to \mathbb{P}^2 \qquad \dim R_{2,k} = \binom{2+k}{2} = \dfrac{(k+2)(k+1)}{2} = \dfrac{k^2}{2} + \dfrac{3k}{2} + 1$

k	0	1	2	3	4	5	6
$\dim R_{2,k}$	1	3	6	10	15	21	28
$\ell(kD)$	1	3	6	9	12	15	18
diff	0	0	0	1	3	6	10

Thus there is a cubic poly $F_3(x_0, x_1, x_2)$ that vanishes on $\phi_{3p}(X)$

In fact $\mathcal{I} = \ker\left(R_2 \to R(X, 3p)\right)$ is generated as an ideal by F_3 : $\mathcal{I} = R_2 \cdot (F_3)$ and

$$R(X, 3p) \cong R_2/\mathcal{I} = \mathbb{C}[x_0, x_1, x_2]/F_3$$

For $g=3$, $D=K$, X not hyperelliptic, the same reasoning shows there is a quartic poly $F_4(x_0, x_1, x_2)$ that vanishes on $\phi_K(X)$.